Structural Analysis and Output Feedback Control of Nonlinear Multivariable Processes

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This article concerns general multivariable nonlinear processes, particularly those with singular characteristic matrix. A precise characterization of the structural process properties that give rise to generic singularity of the characteristic matrix is initially developed within a graph-theoretic analysis framework. An output feedback controller synthesis problem is then formulated for multivariable processes with singular characteristic matrix. A comprehensive solution to this problem is obtained through a combination of dynamic state feedback controllers and state observers. The performance and robustness characteristics of the proposed control methodology are illustrated through simulations in a double-effect evaporator.

Introduction

The development of general and practical nonlinear multivariable control methods is a well-recognized need, both by industry and academia (for example, Shinskey, 1988), and has motivated a flourishing research activity in recent years. On one hand, numerical methods in an operator inversion framework (Economou et al., 1986, Parrish and Brosilow, 1988) and on-line optimization methods in a model predictive framework (for example, Biegler and Rawlings, 1991) have been proposed for addressing the nonlinear control problem. On the other hand, differential geometric methods have provided an analytical synthesis framework for this purpose (for example, the tutorial articles by Kravaris and Kantor, 1990a,b; and the paper by Kravaris and Arkun, 1991).

Within the geometric framework, and for Single-Input Single-Output (SISO) minimum-phase processes, output feedback controllers have been developed that modify the process dynamics to achieve a desired input/output behavior (Daoutidis and Kravaris, 1992a) and allow for feedforward compensation of disturbances (Daoutidis and Alhumaizi, 1993). For Multiple-Input Multiple-Output (MIMO) nonlinear processes, research has primarily focused on the class of processes that can be decoupled via static state feedback (Freund, 1975; Ha and Gilbert, 1986; Kravaris and Soroush, 1990; Daoutidis et al., 1990; Daoutidis and Kravaris, 1994). These processes are characterized by certain constraints in the process model structure, expressed through a nonsingularity condition on the so-called characteristic (or decoupling) matrix. Whenever this condition is satisfied, the generalization of most of the SISO controller synthesis results in a MIMO setting is straightforward conceptually.

However, for typical MIMO processes (for example, separation processes and chemical reactors), the condition of nonsingular characteristic matrix is violated generically, giving rise to a fundamentally different control problem: dynamic state feedback has to be employed to induce a desired input/output behavior, in the general case. The study of such processes has mainly addressed geometric, system theoretic properties (Isidori and Moog, 1988; Isidori, 1989). The available controller synthesis results are limited to the problem of input/output decoupling via dynamic state feedback (for example, Descusse and Moog, 1985, 1987; Nijmeijer and Respondek, 1988). However, requesting decoupling in the closed-loop system imposes a structural constraint which may lead to unacceptable performance or stability characteristics (Isidori and Grizzle, 1988). Moreover, measurements of the state variables are required for the implementation of the control laws, which may not be possible or economically justifiable.

Motivated by the above, this article focuses on general continuous-time MIMO nonlinear processes, and particularly those with singular characteristic matrix. The purpose of this article is then two-fold:

- To analyze the underlying structure of such processes, both from a mathematical and a physical point of view.
- To develop an explicit output feedback controller synthesis framework for such processes.

To this end, a review of some fundamental concepts from differential geometry will be given initially. Then, a precise characterization of the structural process properties that give rise to generic singularity of the characteristic matrix will be developed, within a graph-theoretic analysis framework. This will allow connecting an abstract rank deficiency condition with concrete physical characteristics and specific classes of chemical processes. Representative chemical reactor examples will be given to illustrate the application of the theoretical analysis results. In the next section, an output feedback controller synthesis problem will be formulated for the class of processes under consideration. Initially, a related dynamic state feedback synthesis problem will be solved. Then, combination of dynamic state feedback and state observers will be employed for the synthesis of state-space realizations of output feedback controllers that achieve the design objectives in the closed-loop system. Finally, the developed control methodology will be applied to a double-effect evaporator and its performance and robustness characteristics will be evaluated through simulations.

Preliminaries

We will consider square MIMO nonlinear processes with a state-space representation of the form:

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x)u_j$$
$$y_i = h_i(x),\ i = 1, \ldots, m \tag{1}$$

where x denotes the vector of state variables, u_j denotes a manipulated input, and y_i denotes an output (to be controlled). In the above description, it is assumed that the input variables represent deviations from their nominal values. It is also assumed that $x \in X \subset \mathbb{R}^n$, where X is open and connected, $u = [u_1, \ldots, u_m]^T \in \mathbb{R}^m$, and $y = [y_1, \ldots, y_m]^T \in \mathbb{R}^m$. Finally, $f(x)$, $g_j(x)$ are used to denote analytic vector fields on X, and $h_i(x)$ to denote analytic scalar fields on X. In a more compact vector notation, Eq. 1 can take the form:

$$\dot{x} = f(x) + g(x)u$$
$$y_i = h_i(x),\ i = 1, \ldots, m \tag{2}$$

where $g(x)$ is a $(n \times m)$ matrix with columns the vector fields $g_1(x), \ldots, g_m(x)$.

In what follows, we will be using the standard Lie derivative notation

$$L_f\, h_i(x) = \sum_{l=1}^{n} \frac{\partial h_i(x)}{\partial x_l} f_l(x)$$

where $f_l(x)$ denotes the lth row element of $f(x)$. One can define higher order Lie derivatives $L_f^k h_i(x) = L_f L_f^{k-1} h_i(x)$ as well as mixed Lie derivatives $L_{g_j} L_f^{k-1} h_i(x)$ in an obvious way.

We begin with some preliminary definitions. For the MIMO nonlinear process described by Eq. 1, let r_{ij} denote the relative order of the output y_i with respect to the manipulated input u_j, that is, the smallest integer for which:

$$L_{g_j} L_f^{r_{ij}-1} h_i(x) \not\equiv 0 \tag{3}$$

If such an integer does not exist, we say that $r_{ij} = \infty$.

The relative order r_{ij} represents the number of integrations that u_j has to go through before it affects y_i, and, hence, can be interpreted as a measure of how direct effect u_j has on y_i (Daoutidis and Kravaris, 1992b).

For the system of Eq. 1, let M_r denote the matrix of the relative orders r_{ij} (Daoutidis and Kravaris, 1992b):

$$M_r = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & & \vdots \\ r_{m1} & \cdots & r_{mm} \end{bmatrix} \tag{4}$$

Finally, let r_i denote the relative order of the output y_i with respect to the manipulated input vector u, defined as:

$$r_i = \min(r_{i1}, \ldots, r_{im}) \tag{5}$$

An immediate consequence of the definition of the relative orders r_i is the following expressions for the derivatives of the outputs y_i, for $i = 1, \ldots, m$:

$$\frac{d^k y_i}{dt^k} = L_f^k h_i(x), \quad k = 0, \ldots, r_i - 1$$
$$\frac{d^{r_i} y_i}{dt^{r_i}} = L_f^{r_i} h_i(x) + \sum_{j=1}^{m} u_j L_{g_j} L_f^{r_i-1} h_i(x) \tag{6}$$

It is assumed that a finite relative order r_i exists for every i, since this is a necessary condition for output controllability. Then, the following relation holds:

$$\begin{bmatrix} \frac{d^{r_1} y_1}{dt^{r_1}} \\ \vdots \\ \frac{d^{r_m} y_m}{dt^{r_m}} \end{bmatrix} = \begin{bmatrix} L_f^{r_1} h_1(x) \\ \vdots \\ L_f^{r_m} h_m(x) \end{bmatrix} + \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix} u \tag{7}$$

The matrix:

$$C(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix} \tag{8}$$

is called the *characteristic matrix* of the system (Claude, 1986).

Assuming nonsingularity of the characteristic matrix $C(x)$ (that is, $\det C(x) \neq 0$, $\forall x \in X$, or equivalently, rank $C(x) = m$ for $x \in X$) allows to explicitly calculate:

1. The process inverse dynamics and zero dynamics (see for example, Daoutidis and Kravaris, 1991).
2. Dynamic output feedback controller realizations that achieve a desired input/output behavior, based on combination of static state feedback laws and state observers (Daoutidis and Kravaris, 1994).

In general, the analysis and controller synthesis for processes with nonsingular characteristic matrix is conceptually similar to the one for SISO nonlinear processes.

In this work we will focus on the more general class of MIMO nonlinear processes for which rank $C(x) = m_1 < m$ for $x \in X$. Such a singularity in the characteristic matrix usually implies a strong coupling between manipulated inputs and controlled outputs (in a sense that will be made precise later), and hence, a truly multivariable nature of the associated control problem.

Structural Analysis of Multivariable Nonlinear Processes

From an analysis point of view, it is important to characterize the pattern of process variable interactions that give rise to generic singularity of the characteristic matrix. This will allow to associate an abstract rank deficiency condition with physical characteristics and specific classes of chemical processes, providing, thus, some physical insight on the nature of such singularity. To this end, let us first review the notion of a structural matrix and its generic rank (for example, Daoutidis and Kravaris, 1992b).

A *structural matrix* is a matrix having fixed zeros in certain locations and arbitrary entries in the remaining locations. For a given matrix, its *equivalent structural matrix* is the one which has zeros and arbitrary entries in exactly the same locations as the zeros and the nonzero entries of the original matrix. The *generic rank of a structural matrix* is the maximal rank that the matrix achieves as a function of its arbitrary nonzero elements. A matrix will be called *structurally nonsingular* if its equivalent structural matrix has full generic rank, while it will be called *structurally singular* if its equivalent structural matrix has generic rank deficiency. Let us also recall the following theorem from Daoutidis and Kravaris (1992b):

Theorem 1. Consider a nonlinear system in the form of Eq. 1 and its characteristic matrix $C(x)$. Then, the generic rank of the structural matrix which is equivalent to $C(x)$ is equal to m if and only if the output variables y_i can be rearranged so that the minimum relative order in each row of the relative order matrix appears in the major diagonal position, that is, so that M_r takes the form:

$$M_r = \begin{bmatrix} r_1 & r_{12} & \cdots & r_{1m} \\ r_{21} & r_2 & \cdots & r_{2m} \\ \vdots & \vdots & & \vdots \\ r_{m1} & r_{m2} & \cdots & r_m \end{bmatrix} \qquad (9)$$

Theorem 1 provides some insight on the occurrence of structural singularity of the characteristic matrix. In particular, it suggests that structural singularity of the characteristic matrix

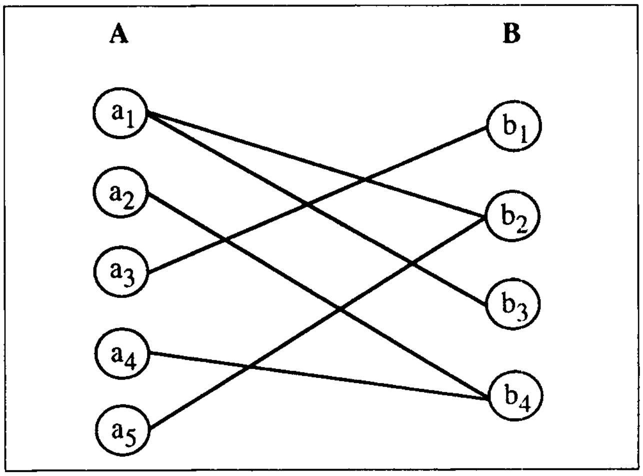

Figure 1. Example of a bipartite graph.

arises due to a *strong coupling* among the input and output variables that does not allow to identify for each output, a distinct input that affects this output at least as directly (in the sense of number of integrations) as the other inputs.

In what follows, the conditions that give rise to structural singularity of the characteristic matrix will be characterized precisely in a graph-theoretic setting, leading to a transparent interpretation of the nature of such singularity. Let us first review some fundamental concepts from graph theory (for example, Ore, 1962; Tucker, 1984 for details), necessary for this purpose.

A *bipartite graph* $G(A, B, E)$ is defined as two sets of vertices A, B and a set of edges E such that any edge $e \in E$ is of the form (a, b) where $a \in A$ and $b \in B$, that is, there is no edge with both endpoints in the same set. An example of a bipartite graph is shown in Figure 1. In a bipartite graph $G(A, B, E)$, two elements a, b of the sets A, B are called *adjacent* if $(a, b) \in E$. Finally, the symbol $|A|$ will be used to denote the *cardinality* of a set A, that is, the number of elements in the set A.

We can now state the main result of this section in the form of a theorem. The proof can be found in the Appendix.

Theorem 2. Consider a nonlinear system in the form of Eq. 1, its characteristic matrix $C(x)$, and the bipartite graph $G(U, Y, E)$, where $U = \{u_1, \ldots, u_m\}$, $Y = \{y_1, \ldots, y_m\}$ and $(u_j, y_i) \in E$ if $r_{ij} = r_i$. Then, the following statements are equivalent:

1. The generic rank of the structural matrix which is equivalent to $C(x)$ is equal to m.
2. For each subset B of Y the following relation holds:

$$|R(B)| \geq |B| \qquad (10)$$

where $R(B)$ is the set of vertices in the set U which are adjacent to vertices in B.

According to theorem 2, the characteristic matrix of a system of the form of Eq. 1 will be structurally singular if and only if there exist *some* outputs, such that the number of inputs affecting these outputs "most directly" is smaller than the number of these outputs.

A special case of singularity of $C(x)$, which is quite common in the case of interconnected units and staged processes, is when one (or more) of its columns is identically equal to 0. In the setting of theorem 2, this type of singularity arises because *all m* outputs are affected "most directly" by $m-1$ (or less, in the case of multiple zero columns) inputs.

Remark 1. The result of theorem 2 can be generalized to nonsquare systems in a straightforward way.

Remark 2. Theorems 1 and 2 provide necessary and sufficient conditions for singularity of the structural matrix equivalent to $C(x)$, and, hence, sufficient conditions for generic singularity of $C(x)$ itself. Clearly, the characteristic matrix $C(x)$ can be generically singular, even if it is structurally nonsingular. In this case, the singularity is due to the specific form of the functions f, g, h, and not the pattern of process variable interactions.

In what follows we will give some physical examples in order to illustrate the interpretation of structural singularity of the characteristic matrix in the framework of Theorem 2, and establish connections with specific classes of chemical processes.

Example 1

Consider the cascade of two CSTRs in series shown in Figure 2. A reactant stream at flow rate F, molar concentration C_{Ao} and temperature T_o enters the first reactor, where the elementary reaction:

$$A \rightarrow B$$

takes place. The outlet stream from the first reactor, at concentration C_{A1} and temperature T_1, is fed to the second reactor. The product stream leaves the second reactor at concentration C_{A2} and temperature T_2. It is desired to control T_2 and C_{A2} in the product stream using the two heat inputs to the reactors Q_1 and Q_2 as manipulated inputs.

Assuming that the liquid holdups V_1 and V_2, the heat of reaction ΔH_r, the liquid density ρ, and the liquid specific heat capacity C_p, are constant, the process dynamics is described by the following set of equations:

$$\frac{dC_{A1}}{dt}=\frac{F}{V_1}(C_{Ao}-C_{A1})-kC_{A1}\exp\left(\frac{-E}{RT_1}\right)$$

$$\frac{dT_1}{dt}=\frac{F}{V_1}(T_o-T_1)+\frac{Q_1}{\rho V_1 C_p}+\frac{(-\Delta H_r)}{\rho C_p}kC_{A1}\exp\left(\frac{-E}{RT_1}\right)$$

$$\frac{dC_{A2}}{dt}=\frac{F}{V_2}(C_{A1}-C_{A2})-kC_{A2}\exp\left(\frac{-E}{RT_2}\right)$$

$$\frac{dT_2}{dt}=\frac{F}{V_2}(T_1-T_2)+\frac{Q_2}{\rho V_2 C_p}+\frac{(-\Delta H_r)}{\rho C_p}kC_{A2}\exp\left(\frac{-E}{RT_2}\right) \tag{11}$$

Defining the state variables:

$$x_1=C_{A1},\ x_2=T_1,\ x_3=C_{A2},\ x_4=T_2$$

the controlled outputs,

$$y_1=C_{A2},\ y_2=T_2$$

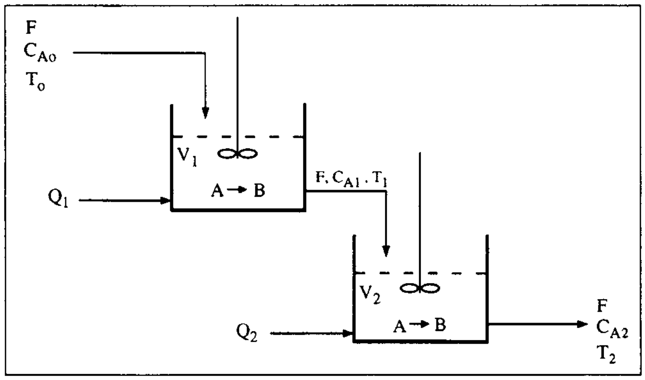

Figure 2. Cascade of two CSTRs in series.

and the manipulated inputs

$$u_1=(Q_1-Q_{1s}),\ u_2=(Q_2-Q_{2s})$$

where the subscript s denotes the steady-state values, the following state-space representation of the process is obtained:

$$\begin{aligned}\dot{x}_1&=f_1(x_1,x_2)\\ \dot{x}_2&=f_2(x_1,x_2)+g_{12}u_1\\ \dot{x}_3&=f_3(x_1,x_3,x_4)\\ \dot{x}_4&=f_4(x_2,x_3,x_4)+g_{24}u_2\\ h_1(x)&=x_3\\ h_2(x)&=x_4\end{aligned} \tag{12}$$

where:

$$f(x)=\begin{bmatrix}\frac{F}{V_1}(C_{Ao}-x_1)-kx_1\exp\left(\frac{-E}{Rx_2}\right)\\ \frac{F}{V_1}(T_o-x_2)+\frac{(-\Delta H_r)}{\rho C_p}kx_1\exp\left(\frac{-E}{Rx_2}\right)+\frac{Q_{1s}}{\rho V_1 C_p}\\ \frac{F}{V_2}(x_1-x_3)-kx_3\exp\left(\frac{-E}{Rx_4}\right)\\ \frac{F}{V_2}(x_2-x_4)+\frac{(-\Delta H_r)}{\rho C_p}kx_3\exp\left(\frac{-E}{Rx_4}\right)+\frac{Q_{2s}}{\rho V_2 C_p}\end{bmatrix}$$

$$g_1(x)=\begin{bmatrix}0\\ \frac{1}{\rho V_1 C_p}\\ 0\\ 0\end{bmatrix},\ g_2(x)=\begin{bmatrix}0\\ 0\\ 0\\ \frac{1}{\rho V_2 C_p}\end{bmatrix}$$

and $g_{ij}(x)$ is the jth component of the vector field $g_i(x)$. A straightforward calculation of the relative orders yields:

$$M_r=\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix}$$

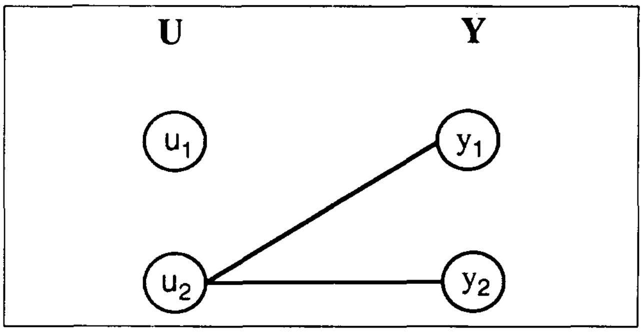

Figure 3. Bipartite graph $G(U, Y, E)$ for the system in example 1.

for the relative order matrix, while the corresponding characteristic matrix is:

$$C(x)=\begin{bmatrix} 0 & -\left(\frac{E}{Rx_4^2}\right)kx_3\exp\left(\frac{-E}{Rx_4}\right)\frac{1}{\rho V_2 C_p} \\ 0 & \frac{1}{\rho V_2 C_p} \end{bmatrix}$$

Clearly, the characteristic matrix is structurally singular (and hence, generically singular) because of the zero first column. The bipartite graph $G(U,Y,E)$ defined in theorem 2, for the above system includes the two vertex sets $U=\{u_1, u_2\}$, $Y=\{y_1, y_2\}$ and the edge set $E=\{(u_2, y_1), (u_2, y_2)\}$, as shown in Figure 3. Consider now the set $B=Y$ with $|B|=2$. The corresponding set of adjacent vertices in U, $R(B)=\{u_2\}$ has cardinality $|R(B)|=1<2$, resulting thus in the structural singularity of $C(x)$ according to theorem 2. In physical terms, the structural singularity of $C(x)$ arises because both the controlled outputs y_1 and y_2 are affected most directly by the single manipulated input u_2.

Example 2

Consider the CSTR shown in Figure 4, where the irreversible second-order reaction:

$$A+B\rightarrow C+D$$

takes place. The reactants A, B are fed to the reactor in two feed streams, at flow rates F_A and F_B, molar concentrations C_{Ao} and C_{Bo} and temperatures T_A and T_B, respectively. The product stream leaves the reactor at a flow rate F, molar concentrations C_A, C_B, C_C, C_D, and temperature T. Q is the heat input to the reactor. The objective is to control C_C, C_A, T, and V using F_A, T_B, Q, and F as the manipulated inputs.

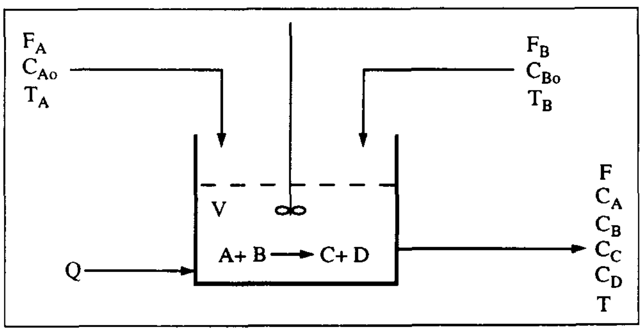

Figure 4. CSTR with irreversible reaction $A+B\rightarrow C+D$.

Assuming ρ, the density of the liquid, C_p, the specific heat capacity of the liquid, and ΔH_r, the heat of reaction, to be constant, and the reaction rate to be of the form:

$$-r_A = kC_AC_B\exp\left(\frac{-E}{RT}\right)$$

the process model is given by the following set of differential equations:

$$\frac{dV}{dt}=F_A+F_B-F$$

$$\frac{dC_A}{dt}=\frac{F_A}{V}(C_{Ao}-C_A)-\frac{F_B}{V}C_A-kC_AC_B\exp\left(\frac{-E}{RT}\right)$$

$$\frac{dC_B}{dt}=-\frac{F_A}{V}C_B+\frac{F_B}{V}(C_{Bo}-C_B)-kC_AC_B\exp\left(\frac{-E}{RT}\right)$$

$$\frac{dC_C}{dt}=-\frac{F_A}{V}C_C-\frac{F_B}{V}C_C+kC_AC_B\exp\left(\frac{-E}{RT}\right)$$

$$\frac{dT}{dt}=\frac{F_A}{V}(T_A-T)+\frac{F_B}{V}(T_B-T)+\frac{Q}{\rho VC_p}+\frac{(-\Delta H_r)}{\rho C_p}kC_AC_B\exp\left(\frac{-E}{RT}\right) \quad (13)$$

Defining the state variables,

$$x_1=V,\quad x_2=C_A,\quad x_3=C_B,\quad x_4=C_C,\quad x_5=T$$

the output variables,

$$y_1=C_C,\quad y_2=C_A,\quad y_3=T,\quad y_4=V$$

and the manipulated inputs,

$$u_1=(F_A-F_{As}),\quad u_2=(T_B-T_{Bs}),$$
$$u_3=(Q-Q_s),\quad u_4=(F-F_s)$$

where the subscript s denotes the steady-state values, one obtains the following state-space representation of the process:

$$\dot{x}_1=f_1+g_{11}u_1+g_{41}u_4$$

$$\dot{x}_2=f_2(x_1,x_2,x_3,x_5)+g_{12}(x_1,x_2)u_1$$

$$\dot{x}_3=f_3(x_1,x_2,x_3,x_5)+g_{13}(x_1,x_3)u_1$$

$$\dot{x}_4=f_4(x_1,x_2,x_3,x_4,x_5)+g_{14}(x_1,x_4)u_1$$

$$\dot{x}_5=f_5(x_1,x_2,x_3,x_5)+g_{15}(x_1,x_5)u_1+g_{25}(x_1)u_2+g_{35}(x_1)u_3$$

$$
\begin{aligned}
h_1(x) &= x_4 \\
h_2(x) &= x_2 \\
h_3(x) &= x_5 \\
h_4(x) &= x_1
\end{aligned}
\qquad (14)
$$

where

$$
f(x) = \begin{bmatrix}
F_B - F_s + F_{As} \\
-\frac{F_B x_2}{x_1} - k x_2 x_3 \exp\left(\frac{-E}{R x_5}\right) + \frac{F_{As}}{x_1}(C_{Ao} - x_2) \\
\frac{F_B}{x_1}(C_{Bo} - x_3) - k x_2 x_3 \exp\left(\frac{-E}{R x_5}\right) - \frac{F_{As} x_3}{x_1} \\
-\frac{F_B x_4}{x_1} + k x_2 x_3 \exp\left(\frac{-E}{R x_5}\right) - \frac{F_{As} x_4}{x_1} \\
-\frac{F_B x_5}{x_1} + \frac{(-\Delta H_r)}{\rho C_p} k x_2 x_3 \exp\left(\frac{-E}{R x_5}\right) + \frac{F_{As}}{x_1}(T_A - x_5) + \frac{F_B T_{Bs}}{x_1} + \frac{Q_s}{\rho C_p x_1}
\end{bmatrix}
$$

$$
g_1(x) = \begin{bmatrix} 1 \\ \frac{(C_{Ao} - x_2)}{x_1} \\ -\frac{x_3}{x_1} \\ -\frac{x_4}{x_1} \\ \frac{(T_A - x_5)}{x_1} \end{bmatrix}, \quad
g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{F_B}{x_1} \end{bmatrix},
$$

$$
g_3(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\rho C_p x_1} \end{bmatrix}, \quad
g_4(x) = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

and g_{ij} denotes the jth component of the vector field g_i. It can be easily verified that the relative order matrix for the system is:

$$
M_r = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & \infty & \infty & 1 \end{bmatrix}
$$

and the corresponding characteristic matrix is:

$$
C(x) = \begin{bmatrix}
-\frac{x_4}{x_1} & 0 & 0 & 0 \\
\frac{(C_{Ao} - x_2)}{x_1} & 0 & 0 & 0 \\
\frac{(T_A - x_5)}{x_1} & \frac{F_B}{x_1} & \frac{1}{\rho C_p x_1} & 0 \\
1 & 0 & 0 & -1
\end{bmatrix}
$$

As described in theorem 2, the bipartite graph $G(U, Y, E)$ for the system is shown in Figure 5, with the two vertex sets $U = \{u_1, u_2, u_3, u_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$, and the edge set:

$$
E = \{(u_1, y_1), (u_1, y_2), (u_1, y_3), (u_1, y_4), (u_2, y_3), (u_3, y_3), (u_4, y_4)\}
$$

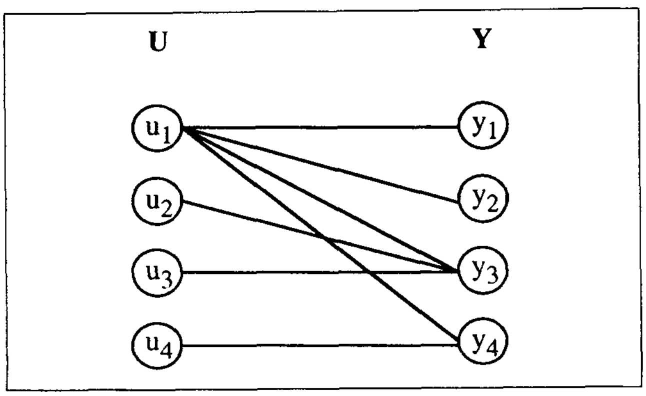

Figure 5. Bipartite graph *G(U, Y, E)* for the system in example 2.

Clearly, considering the set $B = \{y_1, y_2\} \subset Y$ with $|B| = 2$, the corresponding set of adjacent vertices $R(B) = \{u_1\}$ has cardinality $|R(B| = 1 < 2$, resulting in the structural singularity of $C(x)$. In physical terms, the outputs y_1 and y_2 are affected most directly by only one manipulated input, u_1.

We can now proceed with the formulation and solution of a general output feedback synthesis problems for general multivariable nonlinear processes. As will become apparent, the results that will be developed will incorporate the existing controller synthesis results for processes with nonsingular characteristic matrix.

Formulation of the Output Feedback Synthesis Problem

Referring to systems of the form of Eq. 1, with rank $C(x) = m_1 < m$, we wish to synthesize a dynamic compensator that uses measurements of the output variables and the output set points, to enforce an input/output behavior of the form:

$$y_1 + \sum_{i=1}^{m} \sum_{k=1}^{s_i} \gamma_{ik}^1 \frac{d^k y_i}{dt^k} = y_{sp1}$$

$$y_2 + \sum_{i=1}^{m} \sum_{k=1}^{s_i} \gamma_{ik}^2 \frac{d^k y_i}{dt^k} = y_{sp2}$$

$$\vdots$$

$$y_m + \sum_{i=1}^{m} \sum_{k=1}^{s_i} \gamma_{ik}^m \frac{d^k y_i}{dt^k} = y_{spm} \tag{15}$$

where γ_{ik}^j are adjustable constant parameters, with:

$$\det \begin{bmatrix} \gamma_{1s_1}^1 & \gamma_{2s_2}^1 & \cdots & \gamma_{ms_m}^1 \\ \gamma_{1s_1}^2 & \gamma_{2s_2}^2 & \cdots & \gamma_{ms_m}^2 \\ \vdots & \vdots & & \vdots \\ \gamma_{1s_1}^m & \gamma_{2s_2}^m & \cdots & \gamma_{ms_m}^m \end{bmatrix} \neq 0, \tag{16}$$

$y_{sp1}, \ldots, y_{spm}$ are the output set points, and s_i are appropriate indices which determine the order of the closed-loop response. In a more compact vector form, Eq. 15 can take the form:

$$y + \sum_{i=1}^{m} \sum_{k=1}^{s_i} \gamma_{ik} \frac{d^k y_i}{dt^k} = y_{sp} \tag{17}$$

where $\gamma_{ik} = [\gamma_{ik}^1 \ldots \gamma_{ik}^m]^T$. Furthermore, we would like the compensator to guarantee closed-loop input/output and internal stability, and rejection of disturbances and modeling errors.

Note the requested closed-loop input/output behavior, which is of finite dimension and does not possess any finite zeros. For convenience, a linear input/output behavior is requested with a static gain matrix equal to the identity matrix. Input/output stability and performance characteristics can then be transparently incorporated in the design procedure, through the choice of the adjustable parameters. Appropriate choice of the adjustable parameters will also allow achieving any degree of input/output coupling in the closed-loop system. A common choice in practical applications is the requirement of an input/output decoupled closed-loop system. In this case, the postulated input/output behavior in the synthesis problem takes the simplified form:

$$y_1 + \sum_{k=1}^{s_1} \gamma_{1k}^1 \frac{d^k y_1}{dt^k} = y_{sp1}$$

$$y_2 + \sum_{k=1}^{s_2} \gamma_{2k}^2 \frac{d^k y_2}{dt^k} = y_{sp2}$$

$$\vdots$$

$$y_m + \sum_{k=1}^{s_m} \gamma_{mk}^m \frac{d^k y_m}{dt^k} = y_{spm} \tag{18}$$

or in more compact notation:

$$y_i + \sum_{k=1}^{s_i} \gamma_{ik}^i \frac{d^k y_i}{dt^k} = y_{spi} \tag{19}$$

for $i = 1, \ldots, m$.

Output Feedback Controller Synthesis Methodology

The output feedback controller synthesis problem formulated in the previous section will be addressed in a state-space framework. The natural first step to this end is the development of state feedback controllers that solve a related synthesis problem; one can then address the output feedback control problem through the combination of the state feedback controllers with appropriate state observers.

To motivate the approach that we will follow, let's recall Eq. 7 that provides an explicit description of the input/output behavior for the system of Eq. 1. Referring to this equation, it is clear that whenever rank $C(x) = m$, an appropriate choice of the manipulated inputs as a function of the state variables x, that is, an appropriate *static* state feedback law, can cancel the nonlinearity in the input/output map and induce a desired input/output behavior. On the other hand, in the general case that rank $C(x) < m$, because of the singularity in the input/output map of Eq. 7, static state feedback will no longer suffice to induce a desired input/output behavior. In this case, an explicit and nonsingular description of the input/output behavior can only be obtained between higher than r_ith order derivatives of the outputs and the original manipulated inputs, or, alternatively, the original outputs and some modified manipulated inputs obtained through the addition of integrators in appropriate input channels. Any one of the above two approaches introduces additional state variables, giving rise to *dynamic* state feedback controllers for the modification of the input/output behavior of the process.

Based on the above intuitive considerations, a three-step methodology will be employed towards a systematic and general solution of the posed synthesis problem. The three steps are outlined below:

Step 1. Regular static state feedback preserves the relative orders r_i and the rank of $C(x)$ (Isidori, 1989); for this reason, dynamic state feedback will be initially employed, in order to modify the structural characteristics of the process and obtain a nonsingular input/output relation. In particular, following an algorithmic procedure similar to the dynamic extension algorithm of Descusse and Moog (1985):

- The singularity of the characteristic matrix will be localized in specific input channels.
- The state-space will be extended and the manipulated inputs will be modified, by adding integrators in appropriate input channels.
- Dynamic feedback modification will be employed in the extended state-space in order to achieve a rank increase in the characteristic matrix of the extended system, until its rank becomes equal to m.

Step 2. On the basis of the extended system resulting from step 1, a state feedback law will be derived to induce a desired linear input/output behavior between the process outputs and a set of reference inputs. The dynamic compensator of step 1 and the above control law will be combined, resulting in a dynamic input/output linearizing state feedback compensator.

Step 3. Motivated by the approach for processes with nonsingular characteristic matrix (Daoutidis and Kravaris, 1992a, 1994), the dynamic state feedback compensator of step 2 will be combined with an appropriate state observer and a linear controller with integral action for the synthesis of output feedback controllers that solve the posed synthesis problem (see Figure 6). In the remainder of this section, we will focus on steps 1 and 2, developing a solution to the state feedback synthesis problem, to be used for the output feedback synthesis.

Structural modification of the process dynamics via dynamic state feedback

We begin with a brief review of the algorithmic procedure, which will allow modifying the process structural characteristics and derive a system of increased dimension and modified manipulated inputs which has a nonsingular characteristic matrix. We will follow a formulation similar to the dynamic extension algorithm of Descusse and Moog (1985), using selected manipulated inputs and their derivatives of appropriate order as additional state variables and employing dynamic state feedback compensation. This approach will be preferred over the alternative approach of applying differential operators on the outputs (Hirschorn, 1979; Kravaris and Soroush, 1990), primarily in order to allow expressing performance specifications with respect to the original output variables and not in terms of some redesigned outputs that may have no physical significance.

Initially, we assume that rank $C(x) = m_1 < m$, for $x \in X$.

Iteration 1: Step 1. Calculate an analytic, square and nonsingular matrix $E^{(1)}(x)$, such that $C(x)E^{(1)}(x)$ has its last $m - m_1$ columns identically equal to zero. Such a matrix always exists, although it is not unique, in general (Descusse and Moog, 1985).

Step 2. Because of the assumption of finite r_i for all i and the assumption that rank $C(x) = m_1$, there will be some columns among the first m_1 columns of $C(x)E^{(1)}(x)$ that have two or more elements not identically equal to zero. Assume, without loss of generality, that these are the first l_1 columns of $C(x)E^{(1)}(x)$ (where $0 < l_1 \leq m_1$). Introduce an integrator in series with each one of the input channels corresponding to these l_1 columns and consider the following system in the extended state-space:

$$\dot{x} = f(x) + g(x)\left[\sum_{j=1}^{l_1} e_j^{(1)}(x) z_j + \sum_{j=l_1+1}^{m} e_j^{(1)}(x) v_j^{(1)}\right]$$
$$\dot{z}_1 = v_1^{(1)}$$
$$\dot{z}_2 = v_2^{(1)}$$
$$\vdots$$
$$\dot{z}_{l_1} = v_{l_1}^{(1)} \tag{20}$$

where $e_j^{(1)}$ denotes the jth column of $E^{(1)}$, and $v^{(1)} = [v_1^{(1)}, \ldots, v_m^{(1)}]^T$ denotes a redefined vector of manipulated inputs.

Step 3. Let $x^{(1)} = (x_1, \ldots, x_n, z_1, \ldots, z_{l_1})^T$ denote an extended state vector and consider the system of Eq. 20 in the form:

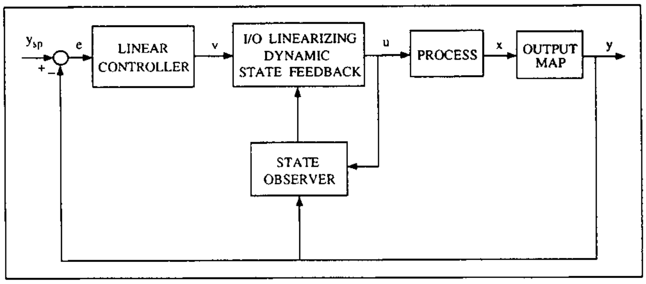

Figure 6. Output feedback synthesis framework.

$$\dot{x}^{(1)} = f^{(1)}(x^{(1)}) + \sum_{j=1}^{m} g_j^{(1)}(x^{(1)}) v_j^{(1)} = f^{(1)}(x^{(1)}) + g^{(1)}(x^{(1)}) v^{(1)} \tag{21}$$

with $x^{(1)}$, $f^{(1)}$, $g^{(1)}$ appropriately defined. Let $C^{(1)}(x^{(1)})$ denote the characteristic matrix of the system of Eq. 21. It can be shown that rank $C^{(1)}(x^{(1)}) = m_2 \geq m_1$ around the nominal equilibrium point (and thus in an appropriately defined open set, denoted by X for simplicity). If $m_2 = m$, then stop. Otherwise go back to Step 1 and resume the procedure for the system of Eq. 21.

The above algorithm will generate a sequence of integers $m_1 \leq m_2 \leq m_3 \leq \ldots$. Under the (verifiable) assumptions of left invertibility in the sense of Hirschorn (Hirschorn, 1979) and strong accessibility (Sussmann and Jurdjevic, 1972), the algorithm will converge in a finite number of iterations (Descusse and Moog, 1985). Suppose that it converges at the κth iteration with a resulting system of the form:

$$\dot{x}^{(\kappa-1)} = f^{(\kappa-1)}(x^{(\kappa-1)}) + g^{(\kappa-1)}(x^{(\kappa-1)})\left[\sum_{j=1}^{l_\kappa} e_j^{(\kappa)}(x^{(\kappa-1)}) z_{l_1+\ldots+l_{\kappa-1}+j} + \sum_{j=l_\kappa+1}^{m} e_j^{(\kappa)}(x^{(\kappa-1)}) v_j^{(\kappa)}\right]$$
$$\dot{z}_{l_1+\ldots+l_{\kappa-1}+1} = v_1^{(\kappa)}$$
$$\vdots$$
$$\dot{z}_{l_1+\ldots+l_{\kappa-1}+l_\kappa} = v_{l_\kappa}^{(\kappa)} \tag{22}$$

or in more compact notation in the extended space:

$$\dot{x}^{(\kappa)} = f^{(\kappa)}(x^{(\kappa)}) + g^{(\kappa)}(x^{(\kappa)}) v^{(\kappa)} \tag{23}$$

where $x^{(\kappa)} = (x_1, \ldots, x_n, z_1, \ldots, z_{l_1}, z_{l_1+1}, \ldots, z_{l_1+l_2}, \ldots, z_{l_1+l_2+\ldots+l_\kappa})^T$ and $f^{(\kappa)}$, $g^{(\kappa)}$ are appropriately defined. Let $r_i^{(\kappa)}$ denote the relative orders of the outputs y_i with respect to $v^{(\kappa)}$, and $C^{(\kappa)}(x^{(\kappa)})$ the corresponding characteristic matrix, with rank equal to m. The above system will now form the basis for the controller synthesis.

Referring to the first iteration of the above algorithm, the following remarks should be made:

(1) In step 1 of the algorithm, the feedback matrix $E^{(1)}(x)$ redefines the manipulated inputs so that the resulting characteristic matrix (that is, the matrix $C(x)E^{(1)}(x)$) has its last $m-m_1$ columns equal to zero. Note that even if the characteristic matrix is singular, but not structurally singular, step 1 induces a structural singularity by appropriately redefining the manipulated inputs.

(2) In step 2 of the algorithm, some of the first m_1 input channels (corresponding to manipulated inputs that appear "too early" when differentiating the outputs) are chosen for the addition of integrators.

(3) The system of Eq. 20 is obtained by applying the following dynamic state feedback compensator to the original system:

$$\begin{aligned} \dot{z}_1 &= v_1^{(1)} \\ \dot{z}_2 &= v_2^{(1)} \\ &\vdots \\ \dot{z}_{l_1} &= v_{l_1}^{(1)} \\ u &= \sum_{j=1}^{l_1} e_j^{(1)}(x)z_j + \sum_{j=l_1+1}^{m} e_j^{(1)}(x)v_j^{(1)} \end{aligned} \tag{24}$$

while at the kth iteration of the algorithm the following dynamic state feedback compensator is applied:

$$\begin{aligned} \dot{z}_{l_1+\dots+l_{k-1}+1} &= v_1^{(k)} \\ &\vdots \\ \dot{z}_{l_1+\dots+l_k} &= v_{l_k}^{(k)} \\ v^{(k-1)} &= \sum_{j=1}^{l_k} e_j^{(k)}(x^{(k-1)})z_{l_1+\dots+l_{k-1}+j} \\ &\quad + \sum_{j=l_k+1}^{m} e_j^{(k)}(x^{(k-1)})v_j^{(k)} \end{aligned} \tag{25}$$

State feedback controller synthesis

We will now address the problem of synthesizing a dynamic state feedback compensator which induces a well-characterized input/output behavior between the process output variables and a set of reference inputs. More specifically, we will request a linear input/output behavior of the form:

$$\sum_{i=1}^{m}\sum_{k=0}^{r_i^{(\kappa)}} \beta_{ik}\frac{d^k y_i}{dt^k} = v \tag{26}$$

where v is the vector of reference inputs and $\beta_{ik}=[\beta_{ik}^1\dots\beta_{ik}^m]^T$ are vectors of adjustable parameters. The basic synthesis result is summarized in theorem 3 that follows (the proof can be found in the Appendix).

Theorem 3. Consider the nonlinear process described by Eq. 1, with rank $C(x)=m_1<m$ and the system of Eq. 23 with rank $C^{(\kappa)}(x^{(\kappa)})=m$ for $x\in X$. Then the dynamic state feedback compensator:

$$\begin{aligned} u &= \sum_{j=1}^{l_1} e_j^{(1)}(x)z_j + \sum_{j=l_1+1}^{m} e_j^{(1)}(x)v_j^{(1)} \\ \dot{z}_1 &= v_1^{(1)} \\ \dot{z}_2 &= v_2^{(1)} \\ &\vdots \\ \dot{z}_{l_1} &= v_{l_1}^{(1)} \\ v^{(1)} &= \sum_{j=1}^{l_2} e_j^{(2)}(x^{(1)})z_{l_1+j} + \sum_{j=l_2+1}^{m} e_j^{(2)}(x^{(1)})v_j^{(2)} \\ \dot{z}_{l_1+1} &= v_1^{(2)} \\ &\vdots \\ \dot{z}_{l_1+l_2} &= v_{l_2}^{(2)} \\ &\vdots \\ v^{(\kappa-1)} &= \sum_{j=1}^{l_\kappa} e_j^{(\kappa)}(x^{(\kappa-1)})z_{l_1+\dots+l_{\kappa-1}+j} \\ &\quad + \sum_{j=l_\kappa+1}^{m} e_j^{(\kappa)}(x^{(\kappa-1)})v_j^{(\kappa)} \\ \dot{z}_{l_1+\dots+l_{\kappa-1}+1} &= v_1^{(\kappa)} \\ &\vdots \\ \dot{z}_{l_1+\dots+l_\kappa} &= v_{l_\kappa}^{(\kappa)} \\ v^{(\kappa)} &= \{[\beta_{1r_1^{(\kappa)}}\dots\beta_{mr_m^{(\kappa)}}]C^{(\kappa)}(x^{(\kappa)})\}^{-1} \\ &\quad \times\left[v-\sum_{i=1}^{m}\sum_{k=0}^{r_i^{(\kappa)}}\beta_{ik}L_{f^{(\kappa)}}^k h_i(x)\right] \end{aligned} \tag{27}$$

where $\beta_{ik}=[\beta_{ik}^1\dots\beta_{ik}^m]^T$ are vectors of adjustable parameters with $\det[\beta_{1r_1}\dots\beta_{mr_m}]\neq 0$, induces an input/output behavior of the form:

$$\sum_{i=1}^{m}\sum_{k=0}^{r_i^{(\kappa)}} \beta_{ik}\frac{d^k y_i}{dt^k} = v$$

where v is a vector of reference inputs.

Remark 3. In the case that we request an input/output decoupled behavior of the form:

$$\sum_{k=0}^{r_i^{(\kappa)}} \beta_{ik}^i\frac{d^k y_i}{dt^k} = v_i \tag{28}$$

for $i=1, \dots, m$, the necessary controller realization can be obtained by simply setting $\beta_{ik}^j=0$ for $i\neq j$ in Eq. 27, in which case the last equation of Eq. 27 takes the simplified form:

$$v^{(\kappa)} = \{\mathrm{diag}[\beta_{ir_i^{(\kappa)}}^i]C^{(\kappa)}(x^{(\kappa)})\}^{-1}\begin{bmatrix} v_1-\sum_{k=0}^{r_1^{(\kappa)}}\beta_{1k}^1 L_{f^{(\kappa)}}^k h_1(x) \\ \vdots \\ v_m-\sum_{k=0}^{r_m^{(\kappa)}}\beta_{mk}^m L_{f^{(\kappa)}}^k h_m(x) \end{bmatrix} \tag{29}$$

Up to this point we outlined the general solution methodology for the posed output feedback synthesis problem, and we developed a dynamic input/output linearizing state feedback synthesis result. In what follows, we will address the output feedback synthesis problem (see step 3 of the output feedback synthesis methodology). To overcome the difficulties associated with the existence and construction of nonlinear state observers, and in analogy with the approach for processes with nonsingular characteristic matrix (Daoutidis and Kravaris, 1992a, 1994), the natural process modes will be used for the state observation. More specifically, in the case of stable process dynamics, an open-loop state observer will be employed, while in the more general case of potential open-loop instability, a reduced-order observer based on the zero-output constrained dynamics (or, simply, the zero dynamics) of the process will be used instead. The above observer choices will allow a comprehensive solution of the output feedback synthesis problem to be developed for processes with stable zero dynamics, with the understanding that alternative state observation schemes can also be incorporated in the proposed synthesis framework.

Output Feedback Controller Synthesis for Open-loop Stable Processes

In the case of processes with stable open-loop dynamics, an output feedback controller that solves the posed synthesis problem can be derived by combining an open-loop state observer (the process dynamic model itself) with the state feedback compensator of theorem 3 and an appropriate linear error feedback controller with integral action. Figure 7 provides a schematic description of the resulting controller structure, while the basic synthesis result is summarized in Theorem 4 that follows.

Theorem 4. Consider the nonlinear process described by Eq. 1, with rank $C(x)=m_1<m$, and the system of Eq. 23 with rank $C^{(\kappa)}(x^{(\kappa)})=m$ for $x\in X$. Then, the dynamic output feedback compensator:

$$
\begin{aligned}
u &= \sum_{j=1}^{l_1} e_j^{(1)}(w)z_j + \sum_{j=l_1+1}^{m} e_j^{(1)}(w)v_j^{(1)}\\
\dot{w} &= f(w)+g(w)\\
&\quad\times\left[\sum_{j=1}^{l_1} e_j^{(1)}(w)z_j + \sum_{j=l_1+1}^{m} e_j^{(1)}(w)v_j^{(1)}\right]\\
\dot{z}_1 &= v_1^{(1)}\\
\dot{z}_2 &= v_2^{(1)}\\
&\vdots\\
\dot{z}_{l_1} &= v_{l_1}^{(1)}\\
&\vdots\\
v^{(\kappa-1)} &= \sum_{j=1}^{l_\kappa} e_j^{(\kappa)}(w^{(\kappa-1)})z_{l_1+\ldots+l_{\kappa-1}+j}\\
&\quad+\sum_{j=l_\kappa+1}^{m} e_j^{(\kappa)}(w^{(\kappa-1)})v_j^{(\kappa)}\\
\dot{z}_{l_1+\ldots+l_{\kappa-1}+1} &= v_1^{(\kappa)}\\
&\vdots\\
\dot{z}_{l_1+\ldots+l_\kappa} &= v_{l_\kappa}^{(\kappa)}
\end{aligned}
$$

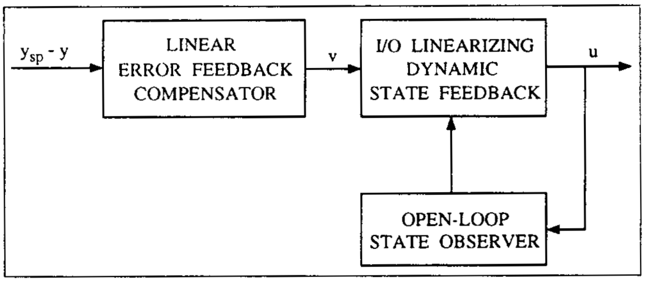

Figure 7. Full-order error feedback controller structure.

$$
\begin{aligned}
v^{(\kappa)} &= \{[\gamma_{1r_1^{(\kappa)}}\ldots\gamma_{mr_m^{(\kappa)}}]C^{(\kappa)}(w^{(\kappa)})\}^{-1}\\
&\quad\times\left\{\sum_{i=1}^{m}\sum_{k=0}^{r_i^{(\kappa)}-1}\beta_{ik}\xi_{k+1}^{(i)}\right.\\
&\quad+[\beta_{1r_1^{(\kappa)}}\ldots\beta_{mr_m^{(\kappa)}}][\gamma_{1r_1^{(\kappa)}}\ldots\gamma_{mr_m^{(\kappa)}}]^{-1}\\
&\quad\times\left[(y_{sp}-y)-\sum_{i=1}^{m}\sum_{k=1}^{r_i^{(\kappa)}-1}\gamma_{ik}\xi_{k+1}^{(i)}\right]\\
&\quad\left.-\sum_{i=1}^{m}\sum_{k=0}^{r_i^{(\kappa)}}\beta_{ik}L_{f^{(\kappa)}}^{k}h_i(w)\right\}\\
\dot{\xi}_1^{(1)} &= \xi_2^{(1)}\\
&\vdots\\
\dot{\xi}_{r_1^{(\kappa)}-1}^{(1)} &= \xi_{r_1^{(\kappa)}}^{(1)}\\
\dot{\xi}_{r_1^{(\kappa)}}^{(1)} &= ([\gamma_{1r_1^{(\kappa)}}\ldots\gamma_{mr_m^{(\kappa)}}]^{-1})_1\\
&\quad\times\left[(y_{sp}-y)-\sum_{i=1}^{m}\sum_{k=1}^{r_i^{(\kappa)}-1}\gamma_{ik}\xi_{k+1}^{(i)}\right]\\
&\vdots\\
\dot{\xi}_1^{(m)} &= \xi_2^{(m)}\\
&\vdots\\
\dot{\xi}_{r_m^{(\kappa)}-1}^{(m)} &= \xi_{r_m^{(\kappa)}}^{(m)}\\
\dot{\xi}_{r_m^{(\kappa)}}^{(m)} &= ([\gamma_{1r_1^{(\kappa)}}\ldots\gamma_{mr_m^{(\kappa)}}]^{-1})_m\\
&\quad\times\left[(y_{sp}-y)-\sum_{i=1}^{m}\sum_{k=1}^{r_i^{(\kappa)}-1}\gamma_{ik}\xi_{k+1}^{(i)}\right]
\end{aligned}
\tag{30}
$$

induces the input/output behavior:

$$y+\sum_{i=1}^{m}\sum_{k=1}^{r_i^{(k)}}\gamma_{ik}\frac{d^k y_i}{dt^k}=y_{sp}$$

The controller realization of theorem 4 was derived by combining the open-loop state observer:

$$\dot{w}=f(w)+g(w)u$$

with the state feedback compensator of theorem 3 to induce the input/output behavior of Eq. 26, and a linear error feedback controller with state space realization:

$$
\begin{aligned}
\dot{\xi}_1^{(1)} &= \xi_2^{(1)} \\
&\vdots \\
\dot{\xi}_{r_1^{(\kappa)}-1}^{(1)} &= \xi_{r_1^{(\kappa)}}^{(1)} \\
\dot{\xi}_{r_1^{(\kappa)}}^{(1)} &= ([\gamma_{1r_1^{(\kappa)}} \cdots \gamma_{mr_m^{(\kappa)}}]^{-1})_1 \left[(y_{sp}-y) - \sum_{i=1}^{m} \sum_{k=1}^{r_i^{(\kappa)}-1} \gamma_{ik}\xi_{k+1}^{(i)} \right] \\
&\vdots \\
\dot{\xi}_1^{(m)} &= \xi_2^{(m)} \\
&\vdots \\
\dot{\xi}_{r_m^{(\kappa)}-1}^{(m)} &= \xi_{r_m^{(\kappa)}}^{(m)} \\
\dot{\xi}_{r_m^{(\kappa)}}^{(m)} &= ([\gamma_{1r_1^{(\kappa)}} \cdots \gamma_{mr_m^{(\kappa)}}]^{-1})_m \left[(y_{sp}-y) - \sum_{i=1}^{m} \sum_{k=1}^{r_i^{(\kappa)}-1} \gamma_{ik}\xi_{k+1}^{(i)} \right]
\end{aligned} \tag{31}
$$

and output map:

$$
\begin{aligned}
v = \sum_{i=1}^{m} \sum_{k=0}^{r_i^{(\kappa)}-1} \beta_{ik}\xi_{k+1}^{(i)} + [\beta_{1r_1^{(\kappa)}} \cdots \beta_{mr_m^{(\kappa)}}][\gamma_{1r_1^{(\kappa)}} \cdots \gamma_{mr_m^{(\kappa)}}]^{-1} \\
\times \left[(y_{sp}-y) - \sum_{i=1}^{m} \sum_{k=1}^{r_i^{(\kappa)}-1} \gamma_{ik}\xi_{k+1}^{(i)} \right]
\end{aligned} \tag{32}
$$

Further details of the proof are omitted for reasons of brevity.

In complete analogy with the results for systems with nonsingular characteristic matrix (for details, see Daoutidis and Kravaris, 1994), appropriate initialization of the states of the controller of Theorem 4 allows the derivation of a reduced-order controller realization, given in Corollary 1 that follows.

Corollary 1. Consider the nonlinear process described by Eq. 1, with rank $C(x) = m_1 < m$, and the system of Eq. 23 with rank $C^{(\kappa)}(x^{(\kappa)}) = m$ for $x \in X$. Then, the dynamic output feedback compensator:

$$
\begin{aligned}
u &= \sum_{j=1}^{l_1} e_j^{(1)}(w) z_j + \sum_{j=l_1+1}^{m} e_j^{(1)}(w) v_j^{(1)} \\
\dot{w} &= f(w) + g(w) \\
&\quad \times \left[\sum_{j=1}^{l_1} e_j^{(1)}(w) z_j + \sum_{j=l_1+1}^{m} e_j^{(1)}(w) v_j^{(1)} \right] \\
\dot{z}_1 &= v_1^{(1)} \\
\dot{z}_2 &= v_2^{(1)} \\
&\vdots \\
\dot{z}_{l_1} &= v_{l_1}^{(1)} \\
&\vdots \\
v^{(\kappa-1)} &= \sum_{j=1}^{l_\kappa} e_j^{(\kappa)}(w^{(\kappa-1)}) z_{l_1+\cdots+l_{\kappa-1}+j} \\
&\quad + \sum_{j=l_\kappa+1}^{m} e_j^{(\kappa)}(w^{(\kappa-1)}) v_j^{(\kappa)} \\
\dot{z}_{l_1+\cdots+l_{\kappa-1}+1} &= v_1^{(\kappa)} \\
&\vdots \\
\dot{z}_{l_1+\cdots+l_\kappa} &= v_{l_\kappa}^{(\kappa)} \\
v^{(\kappa)} &= \{[\gamma_{1r_1^{(\kappa)}} \cdots \gamma_{mr_m^{(\kappa)}}] C^{(\kappa)}(w^{(\kappa)})\}^{-1} \\
&\quad \times \left[(y_{sp}-y) - \sum_{i=1}^{m} \sum_{k=1}^{r_i^{(\kappa)}} \gamma_{ik} L_{f^{(\kappa)}}^{k} h_i(w) \right]
\end{aligned} \tag{33}
$$

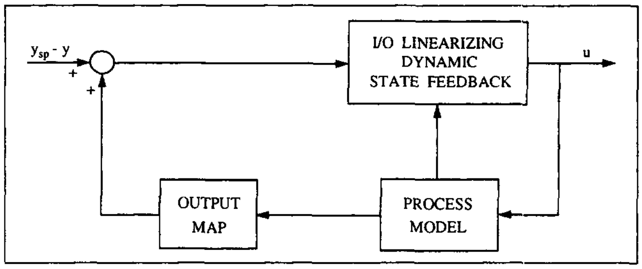

Figure 8. Reduced-order error feedback controller structure.

induces the input/output behavior:

$$
y + \sum_{i=1}^{m} \sum_{k=1}^{r_i^{(k)}} \gamma_{ik} \frac{d^k y_i}{dt^k} = y_{sp}
$$

The structure of the controller realization of Eq. 33 is shown in Figure 8.

Remark 4. In the case that rank $C(x) = m$, and thus $\kappa = 0$, the controllers of Eqs. 30 and 33 reduce to the full- and reduced-order realizations of the error feedback controller derived in Daoutidis and Kravaris (1994) for open-loop stable processes with nonsingular characteristic matrix.

Remark 5. In the case of a postulated input/output decoupled closed-loop behavior of the form of Eq. 19, the necessary controller realizations are obtained by simply setting $\gamma_{ik}^j = 0$ for $i \neq j$ in the controller realizations of Eqs. 30 and 33. Setting $\beta_{ik}^j = 0$ for $i \neq j$ in Eq. 30 allows a further simplification of the controller realization, imposing the additional requirement of a decoupled input/output behavior of the form of Eq. 28 induced by the state feedback controller.

Output Feedback Controller Synthesis for General Processes

In this section, we will focus on processes with potentially unstable open-loop dynamics, but with stable zero dynamics, and we will use an alternative scheme of state reconstruction based on the modes of the zero dynamics. A key difficulty to this end is the derivation of an explicit state space realization of the zero dynamics, which in the case of processes with singular characteristic matrix, entails a quite involved algorithmic procedure (Isidori, 1989). To overcome this difficulty, we will rely on the fact that the dynamic state feedback employed during the structural modification algorithm leaves the zero dynamics of the process unchanged (Isidori, 1989); hence, a state space realization of the zero dynamics will be obtained on the basis of the extended system resulting from the structural modification algorithm and will be used for the state reconstruction scheme. To facilitate the development we will be working in appropriate normal form coordinates. More specifically, consider the extended system of Eq. 23, which under the coordinate transformation:

$$\zeta = \begin{bmatrix} \zeta^{(0)} \\ --- \\ \zeta^{(1)} \\ --- \\ \vdots \\ --- \\ \zeta^{(m)} \end{bmatrix} = T(x^{(\kappa)}) = \begin{bmatrix} t_1(x^{(\kappa)}) \\ \vdots \\ t_{n-\Sigma_i r_i}(x^{(\kappa)}) \\ ------------ \\ h_1(x^{(\kappa)}) \\ L_{f^{(\kappa)}} h_1(x^{(\kappa)}) \\ \vdots \\ L_{f^{(\kappa)}}^{r_1^{(\kappa)}-1} h_1(x^{(\kappa)}) \\ ------------ \\ \vdots \\ ------------ \\ h_m(x^{(\kappa)}) \\ L_{f^{(\kappa)}} h_m(x^{(\kappa)}) \\ \vdots \\ L_{f^{(\kappa)}}^{r_m^{(\kappa)}-1} h_m(x^{(\kappa)}) \end{bmatrix} \quad (34)$$

where $t_1(x^{(\kappa)}), \ldots, t_{n-\Sigma_i r_i^{(\kappa)}}(x^{(\kappa)})$ are scalar fields such that:

$$t_1(x^{(\kappa)}), \ldots, t_{n-\Sigma_i r_i^{(\kappa)}}(x^{(\kappa)}), h_1(x^{(\kappa)}), L_{f^{(\kappa)}} h_1(x^{(\kappa)}), \ldots, L_{f^{(\kappa)}}^{r_1^{(\kappa)}-1} h_1(x^{(\kappa)}), \ldots, h_m(x^{(\kappa)}), L_{f^{(\kappa)}} h_m(x^{(\kappa)}), \ldots, L_{f^{(\kappa)}}^{r_m^{(\kappa)}-1} h_m(x^{(\kappa)})$$

are linearly independent, and $L_{g_j^{(\kappa)}} t_l(x^{(\kappa)}) = 0$ for all l, j, takes the normal form (for details, see Daoutidis and Kravaris, 1991):

$$\begin{aligned}
\dot{\zeta}_1^{(0)} &= F_1(\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(m)}) \\
&\vdots \\
\dot{\zeta}_{n-\Sigma_i r_i^{(\kappa)}}^{(0)} &= F_{n-\Sigma_i r_i^{(\kappa)}}(\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(m)}) \\
\dot{\zeta}_1^{(1)} &= \zeta_2^{(1)} \\
&\vdots \\
\dot{\zeta}_{r_1^{(\kappa)}-1}^{(1)} &= \zeta_{r_1^{(\kappa)}}^{(1)} \\
\dot{\zeta}_{r_1^{(\kappa)}}^{(1)} &= W_1(\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(m)}) + C_1^{(\kappa)}(\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(m)}) v^{(\kappa)} \\
&\vdots \\
\dot{\zeta}_1^{(m)} &= \zeta_2^{(m)} \\
&\vdots \\
\dot{\zeta}_{r_m^{(\kappa)}-1}^{(m)} &= \zeta_{r_m^{(\kappa)}}^{(m)} \\
\dot{\zeta}_{r_m^{(\kappa)}}^{(m)} &= W_m(\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(m)}) + C_m^{(\kappa)}(\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(m)}) v^{(\kappa)} \\
y_1 &= \zeta_1^{(1)} \\
&\vdots \\
y_m &= \zeta_1^{(m)}
\end{aligned} \quad (35)$$

where:

$$F_l(\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(m)}) = [L_{f^{(\kappa)}} t_l(x^{(\kappa)})]_{x^{(\kappa)}=T^{-1}(\zeta)}, \quad l = 1, \ldots, \left(n - \sum_i r_i^{(\kappa)}\right)$$

$$C_i^{(\kappa)}(\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(m)}) = [L_{g^{(\kappa)}} L_{f^{(\kappa)}}^{r_i^{(\kappa)}-1} h_i(x^{(\kappa)})]_{x^{(\kappa)}=T^{-1}(\zeta)}, \quad i = 1, \ldots, m$$

$$W_i(\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(m)}) = [L_{f^{(\kappa)}}^{r_i^{(\kappa)}} h_i(x^{(\kappa)})]_{x^{(\kappa)}=T^{-1}(\zeta)}, \quad i = 1, \ldots, m \quad (36)$$

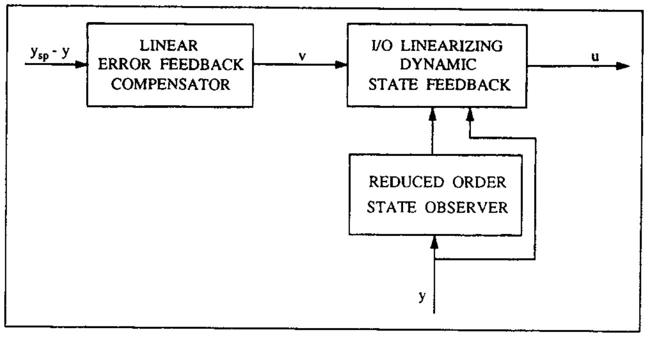

Figure 9. Error and output feedback controller structure.

Referring to the above normal form, let:

$$\mathfrak{D}y_1 = \begin{bmatrix} y_1 \\ \vdots \\ \dfrac{d^{r_1^{(\kappa)}-1} y_1}{dt^{r_1^{(\kappa)}-1}} \end{bmatrix}, \ldots, \mathfrak{D}y_m = \begin{bmatrix} y_m \\ \vdots \\ \dfrac{d^{r_m^{(\kappa)}-1} y_m}{dt^{r_m^{(\kappa)}-1}} \end{bmatrix} \quad (37)$$

Then, according to Daoutidis and Kravaris (1991), the dynamic system:

$$\begin{aligned}
\dot{\zeta}_1^{(0)} &= F_1(\zeta^{(0)}, \mathfrak{D}y_1, \ldots, \mathfrak{D}y_m) \\
&\vdots \\
\dot{\zeta}_{n-\Sigma_i r_i^{(\kappa)}}^{(0)} &= F_{n-\Sigma_i r_i^{(\kappa)}}(\zeta^{(0)}, \mathfrak{D}y_1, \ldots, \mathfrak{D}y_m)
\end{aligned} \quad (38)$$

with $W = [W_1 \ldots W_m]^T$ and $C^{(\kappa)} = [C_1^{(\kappa)} \ldots C_m^{(\kappa)}]^T$ defined in Eq. 36, represents a reduced-order realization for the forced zero dynamics of the system of Eq. 23.

In the above framework of the normal form coordinates, it can be easily seen that the state variables $\zeta^{(1)}$ through $\zeta^{(m)}$ are exactly the vectors $\mathfrak{D}y_1$ through $\mathfrak{D}y_m$, which can be calculated from measurements of the outputs and evaluation of their derivatives. Furthermore, the state variables $\zeta^{(0)}$ can be reconstructed by simulating the forced zero dynamics of Eq. 38, assuming that it is stable. Combining the above state observation scheme with the dynamic state feedback compensator of theorem 3 and the linear compensator of Eqs. 31 and 32, allows the derivation of a state space realization of the output feedback controller that solves the posed synthesis problem. Figure 9 provides a schematic description of the controller structure, while theorem 5 states the main synthesis result (its proof is completely analogous to the one of theorem 4, and is omitted for brevity).

Theorem 5. Consider the nonlinear process described by Eq. 1, with rank $C(x) = m_1 < m$, and the system of Eq. 23 with rank $C^{(\kappa)}(x^{(\kappa)}) = m$ for $x \in X$ in the form of Eq. 35. Then, the dynamic output feedback compensator:

$$\begin{aligned}
u &= \sum_{j=1}^{l_1} e_j^{(1)}(\eta, \mathfrak{D}y_1, \ldots, \mathfrak{D}y_m) z_j + \sum_{j=l_1+1}^{m} e_j^{(1)}(\eta, \mathfrak{D}y_1, \ldots, \mathfrak{D}y_m) v_j^{(1)} \\
\dot{\eta}_1 &= F_1(\eta, \mathfrak{D}y_1, \ldots, \mathfrak{D}y_m) \\
&\vdots \\
\dot{\eta}_{n-\Sigma_i r_i^{(\kappa)}} &= F_{n-\Sigma_i r_i^{(\kappa)}}(\eta, \mathfrak{D}y_1, \ldots, \mathfrak{D}y_m)
\end{aligned}$$

$$\dot{z}_1 = v_1^{(1)}$$
$$\dot{z}_2 = v_2^{(1)}$$
$$\vdots$$
$$\dot{z}_{l_1} = v_{l_1}^{(1)}$$
$$\vdots$$
$$v^{(\kappa-1)} = \sum_{j=1}^{l_k} e_j^{(\kappa)}(\eta, \mathfrak{D}y_1, \ldots, \mathfrak{D}y_m) z_{l_1+\ldots+l_{\kappa-1}+j} + \sum_{j=l_k+1}^{m} e_j^{(\kappa)}(\eta, \mathfrak{D}y_1, \ldots, \mathfrak{D}y_m) v_j^{(\kappa)}$$
$$\dot{z}_{l_1+\ldots+l_{\kappa-1}+1} = v_1^{(\kappa)}$$
$$\vdots$$
$$\dot{z}_{l_1+\ldots+l_\kappa} = v_{l_\kappa}^{(\kappa)}$$
$$v^{(\kappa)} = \{[\beta_{1r_1^{(\kappa)}} \ldots \beta_{mr_m^{(\kappa)}}] C^{(\kappa)}(\eta, \mathfrak{D}y_1, \ldots, \mathfrak{D}y_m)\}^{-1} \times \left[\sum_{i=1}^{m} \sum_{k=0}^{r_i^{(\kappa)}-1} \beta_{ik} \xi_{k+1}^{(i)} + [\beta_{1r_1^{(\kappa)}} \ldots \beta_{mr_m^{(\kappa)}}][\gamma_{1r_1^{(\kappa)}} \ldots \gamma_{mr_m^{(\kappa)}}]^{-1} \times \left[(y_{sp}-y) - \sum_{i=1}^{m} \sum_{k=1}^{r_i^{(\kappa)}-1} \gamma_{ik} \xi_{k+1}^{(i)} \right] - \sum_{i=1}^{m} \sum_{k=0}^{r_i^{(\kappa)}-1} \beta_{ik} \frac{d^k y_i}{dt^k} - \sum_{i=1}^{m} \beta_{ir_i^{(\kappa)}} W_i(\eta, \mathfrak{D}y_1, \ldots, \mathfrak{D}y_m) \right]$$
$$\dot{\xi}_1^{(1)} = \xi_2^{(1)}$$
$$\vdots$$
$$\dot{\xi}_{r_1^{(\kappa)}-1}^{(1)} = \xi_{r_1^{(\kappa)}}^{(1)}$$
$$\dot{\xi}_{r_1^{(\kappa)}}^{(1)} = ([\gamma_{1r_1^{(\kappa)}} \ldots \gamma_{mr_m^{(\kappa)}}]^{-1})_1 \times \left[(y_{sp}-y) - \sum_{i=1}^{m} \sum_{k=1}^{r_i^{(\kappa)}-1} \gamma_{ik} \xi_{k+1}^{(i)} \right]$$
$$\vdots$$
$$\dot{\xi}_1^{(m)} = \xi_2^{(m)}$$
$$\vdots$$
$$\dot{\xi}_{r_m^{(\kappa)}-1}^{(m)} = \xi_{r_m^{(\kappa)}}^{(m)}$$
$$\dot{\xi}_{r_m^{(\kappa)}}^{(m)} = ([\gamma_{1r_1^{(\kappa)}} \ldots \gamma_{mr_m^{(\kappa)}}]^{-1})_m \times \left[(y_{sp}-y) - \sum_{i=1}^{m} \sum_{k=1}^{r_i^{(\kappa)}-1} \gamma_{ik} \xi_{k+1}^{(i)} \right] \quad (39)$$

induces the input/output behavior:

$$y + \sum_{i=1}^{m} \sum_{k=1}^{r_i^{(\kappa)}} \gamma_{ik} \frac{d^k y_i}{dt^k} = y_{sp}$$

Remark 6. In the case that rank $C(x) = m$, that is, $\kappa = 0$, the controller of Eq. 39 reduces exactly to the mixed error and output feedback controller derived by Daoutidis and Kravaris (1994) for minimum-phase processes with nonsingular characteristic matrix.

Remark 7. In the case of a postulated input/output decoupled closed-loop behavior of the form of Eq. 19, the necessary controller realization is obtained by simply setting $\gamma_{ik}^j = 0$ for $i \neq j$ in the controller realization of Eq. 39. Similarly to the case of open-loop stable processes, one could also set $\beta_{ik}^j = 0$ for $i \neq j$, requesting an input/output decoupled behavior of the form of Eq. 28 under the state feedback compensator and employing a cascade of SISO linear compensators with integral action.

Closed-Loop Stability Considerations

The input/output stability of the closed-loop system under the controllers of theorems 4 and 5 will be guaranteed by choosing the adjustable parameters γ_{ij}^k so that the roots of the characteristic equation:

$$\det \begin{bmatrix} (1 + \Sigma_{k=1}^{r_1^{(\kappa)}} \gamma_{1k}^1 s^k) & (\Sigma_{k=1}^{r_2^{(\kappa)}} \gamma_{2k}^1 s^k) & \ldots & (\Sigma_{k=1}^{r_m^{(\kappa)}} \gamma_{mk}^1 s^k) \\ (\Sigma_{k=1}^{r_1^{(\kappa)}} \gamma_{1k}^2 s^k) & (1 + \Sigma_{k=1}^{r_2^{(\kappa)}} \gamma_{2k}^2 s^k) & \ldots & (\Sigma_{k=1}^{r_m^{(\kappa)}} \gamma_{mk}^2 s^k) \\ \vdots & \vdots & & \vdots \\ (\Sigma_{k=1}^{r_1^{(\kappa)}} \gamma_{1k}^m s^k) & (\Sigma_{k=1}^{r_2^{(\kappa)}} \gamma_{2k}^m s^k) & \ldots & (1 + \Sigma_{k=1}^{r_m^{(\kappa)}} \gamma_{mk}^m s^k) \end{bmatrix} = 0 \quad (40)$$

lie in the open left-half plane. The designer has the flexibility to choose γ_{ij}^k in order to satisfy other closed-loop design objectives as well (for example, desirable speed of the response and level of input/output coupling), and at the same time do not violate the constraints in the manipulated inputs.

In addition to input/output stability, the internal stability of the closed-loop system under the controllers of theorems 4 and 5 must also be guaranteed. To this end, Lyapunov's first theorem can be used to show the local internal stability of the closed-loop system under the controller of theorem 4, assuming that the process dynamics and the zero dynamics (Eq. 38) are locally exponential stable and the roots of Eq. 40 and the characteristic equation:

$$\det \begin{bmatrix} (\Sigma_{k=0}^{r_1^{(\kappa)}} \beta_{1k}^1 s^k) & (\Sigma_{k=0}^{r_2^{(\kappa)}} \beta_{2k}^1 s^k) & \ldots & (\Sigma_{k=0}^{r_m^{(\kappa)}} \beta_{mk}^1 s^k) \\ (\Sigma_{k=0}^{r_1^{(\kappa)}} \beta_{1k}^2 s^k) & (\Sigma_{k=0}^{r_2^{(\kappa)}} \beta_{2k}^2 s^k) & \ldots & (\Sigma_{k=0}^{r_m^{(\kappa)}} \beta_{mk}^2 s^k) \\ \vdots & \vdots & & \vdots \\ (\Sigma_{k=0}^{r_1^{(\kappa)}} \beta_{1k}^m s^k) & (\Sigma_{k=0}^{r_2^{(\kappa)}} \beta_{2k}^m s^k) & \ldots & (\Sigma_{k=0}^{r_m^{(\kappa)}} \beta_{mk}^m s^k) \end{bmatrix} = 0 \quad (41)$$

lie in the open left-half plane. Similarly, local internal stability

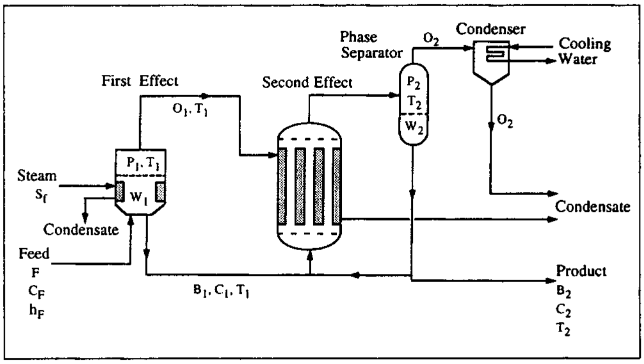

Figure 10. Double-effect evaporator.

of the closed-loop system under the controller of theorem 5 can be shown, relaxing the assumption of local exponential stability of the process dynamics from above.

Application of the Control Methodology to a Double-Effect Evaporator

In what follows, we will illustrate the application of the developed nonlinear control methodology to the double-effect evaporator shown in Figure 10. This process has been the subject of extensive modeling and linear control studies at the University of Alberta (Fisher and Seborg, 1976; Newell, 1971).

Process description and modeling

A solution of triethylene glycol in water is fed to the first effect at flow rate F, solute concentration C_F, and temperature T_F. The solution is concentrated in the first effect using steam at flow rate S_f, generating the overhead vapor stream O_1 and the concentrated bottom stream B_1 with solute concentration C_1. The bottom stream B_1 is fed to the tube side of the second effect at a lower pressure, while the overhead O_1 is fed to the shell side. The concentrated bottom stream B_2, which is the product stream, leaves the second effect with solute concentration C_2. The overhead O_2 from the second effect is condensed and released as condensate. W_1 and W_2 are the liquid holdups, whereas P_1, T_1 and P_2, T_2 are the pressures and temperatures in the first and second effects, respectively.

A standard modeling procedure (for example, Newell, 1971) was followed to develop a dynamic model for the process. More specifically, the following assumptions were made:

(1) The steam chests, tube walls, and so on have negligible heat capacities.

(2) The temperature T_2 is held constant by a sufficiently strong pressure controller on the second effect.

(3) The overhead vapor streams leaving each of the effects have negligible solute concentration compared to the respective bottom liquid streams.

(4) Vapor holdup in each effect is negligible.

Under the above assumptions, total material, solute and enthalpy balances were derived for the two effects. The following expressions were used for the heat inputs Q_1, Q_2 to the two effects:

$$Q_1 = \lambda_S S_f$$

$$Q_2 = U_2 A_2 (T_1 - T_2)$$

$$= O_1 (H_{v1} - h_{c1})$$

where λ_S is the latent heat of condensation of the steam fed to the first effect, U_2 is the overall heat-transfer coefficient in the second effect, A_2 is the total heat-transfer area in the second effect, and H_{v1}, h_{c1} are the specific enthalpies of the vapor and condensate, respectively, given by the empirical correlations (Newell, 1971):

$$h_{c1} = T_1 - 32.0$$

$$H_{v1} = 0.4T_1 + 1{,}066.0$$

On the basis of assumption 2, the following relation for the overhead vapor stream flow rate O_2 was also obtained from the solute and enthalpy balances in the second effect:

$$O_2 = \frac{B_1(h_1 - h_2) - B_1(C_1 - C_2)\dfrac{\partial h_2}{\partial C_2} + Q_2 - L_2}{H_{v2} - h_2 + C_2\dfrac{\partial h_2}{\partial C_2}}$$

where H_{v2} is the specific enthalpy of the vapor at temperature T_2, and h_1, h_2 are the specific enthalpies of the solutions in the two effects, given by the following empirical correlations (Newell, 1971):

$$H_{v2} = 0.4T_2 + 1{,}066.0$$

$$h_1 = T_1(1 - 0.16C_1) - 32.0$$

$$h_2 = T_2(1 - 0.16C_2) - 32.0$$

The above relations were used to arrive at the following fifth-order process dynamic model:

$$\frac{dW_1}{dt} = F - B_1 - \left[\frac{U_2A_2(T_1 - T_2)}{1{,}098 - 0.6T_1}\right]$$

$$\frac{dC_1}{dt} = \frac{F}{W_1}(C_F - C_1) + \frac{C_1}{W_1}\left[\frac{U_2A_2(T_1 - T_2)}{1{,}098 - 0.6T_1}\right]$$

$$\frac{dh_1}{dt} = \frac{F}{W_1}(h_F - h_1) - \frac{L_1}{W_1} + \frac{\lambda_S S_f}{W_1} - \frac{1}{W_1}\left[\frac{U_2A_2(T_1 - T_2)}{1{,}098 - 0.6T_1}\right][1{,}066.1 + 0.4T_1 - h_1]$$

$$\frac{dW_2}{dt} = -\left[\frac{U_2A_2(T_1 - T_2) - L_2}{1{,}098 - 0.6T_2}\right] + \left[1 - \left(\frac{h_1 - T_2 + 32.1 + 0.16T_2C_1}{1{,}098 - 0.6T_2}\right)\right]B_1 - B_2$$

$$\frac{dC_2}{dt}=\frac{C_2}{W_2}\left[\frac{U_2A_2(T_1-T_2)-L_2}{1{,}098-0.6T_2}\right]$$
$$+\left[\frac{C_2}{W_2}\left(\frac{h_1+32.1+0.16T_2C_1-T_2}{1{,}098-0.6T_2}\right)+\left(\frac{C_1-C_2}{W_2}\right)\right]B_1 \quad (42)$$

where h_F is the specific enthalpy of the feed stream, and L_1, L_2 are the heat losses from the two effects. Table 1 provides a detailed description of the process variables and their nominal values. The above model has been shown experimentally to capture the key dynamic characteristics of the process (Newell and Fisher, 1972).

Setting:

$$x_1=W_1,\quad x_2=C_1,\quad x_3=h_1,\quad x_4=W_2,\quad x_5=C_2$$
$$y_1=x_1,\quad y_2=x_4,\quad y_3=x_5$$
$$u_1=(S_f-S_{fs}),\quad u_2=(B_1-B_{1s}),\quad u_3=(B_2-B_{2s})$$

$$g_1(x)=\begin{bmatrix}0\\0\\ \frac{\lambda_S}{x_1}\\0\\0\end{bmatrix};$$

$$g_2(x)=\begin{bmatrix}-1\\0\\0\\1-\left(\frac{x_3-T_2+32.1+0.16T_2x_2}{1{,}098-0.6T_2}\right)\\ \frac{x_5}{x_4}\left[\left(\frac{x_3-T_2+32.1+0.16T_2x_2}{1{,}098-0.6T_2}\right)+\left(\frac{x_2-x_5}{x_5}\right)\right]\end{bmatrix};$$

$$f(x)=\begin{bmatrix}-\frac{U_2A_2(T_1-T_2)}{1{,}098-0.6T_1}-B_{1s}+F_s\\ \frac{x_2}{x_1}\left[\frac{U_2A_2(T_1-T_2)}{1{,}098-0.6T_1}\right]-\frac{x_2}{x_1}F_s+\frac{1}{x_1}F_sC_{Fs}\\ -\frac{1}{x_1}\left[\frac{U_2A_2(T_1-T_2)}{1{,}098-0.6T_1}\right][1{,}066.1+0.4T_1-x_3]-\frac{L_1}{x_1}+\frac{\lambda_S}{x_1}S_{fs}-\frac{x_3}{x_1}F_s+\frac{1}{x_1}F_sh_{Fs}\\ -\left[\frac{U_2A_2(T_1-T_2)-L_2}{1{,}098-0.6T_2}\right]+\left[1-\left(\frac{x_3-T_2+32.1+0.16T_2x_2}{1{,}098-0.6T_2}\right)\right]B_{1s}-B_{2s}\\ \frac{x_5}{x_4}\left[\frac{U_2A_2(T_1-T_2)-L_2}{1{,}098-0.6T_2}\right]+\frac{x_5}{x_4}\left[\left(\frac{x_3-T_2+32.1+0.16T_2x_2}{1{,}098-0.6T_2}\right)+\left(\frac{x_2-x_5}{x_5}\right)\right]B_{1s}\end{bmatrix}$$

where the subscript s denotes the steady-state value, the above equations take the following state-space form:

$$\dot{x}_1=f_1(x_2,x_3)-u_2$$
$$\dot{x}_2=f_2(x_1,x_2,x_3)$$
$$\dot{x}_3=f_3(x_1,x_2,x_3)+g_{13}(x_1)u_1$$
$$\dot{x}_4=f_4(x_2,x_3)+g_{24}(x_2,x_3)u_2-u_3$$
$$\dot{x}_5=f_5(x_2,x_3,x_4,x_5)+g_{25}(x_2,x_3,x_4,x_5)u_2$$
$$y_1=h_1(x)=x_1$$
$$y_2=h_2(x)=x_4$$
$$y_3=h_3(x)=x_5 \quad (43)$$

where

Table 1. Evaporator Variables/Parameters and Their Nominal Values

Variable	Description	Steady-State Value
A_2	Heat-transfer area in 2nd effect (ft^2)	4.6
B_1	Bottoms flow rate in 1st effect (lb/min)	3.3
B_2	Bottoms flow rate in 2nd effect (lb/min)	1.7
C_F	Solute concentration in feed (wt. %)	3.2
C_1	Solute concentration in 1st effect (wt. %)	4.85
C_2	Solute concentration in 2nd effect (wt. %)	9.412
F	Feed flow rate (lb/min)	5.0
h_F	Specific enthalpy of feed (Btu/lb)	162
h_1	Specific enthalpy of liquid in 1st effect (Btu/lb)	194
O_1	Overhead vapor flow from 1st effect (lb/min)	1.7
O_2	Overhead vapor flow from 2nd effect (lb/min)	1.6
P_1	Pressure in first effect (psia)	<25
P_2	Pressure in 2nd effect (psia)	7.5
S_f	Steam flow rate (lb/min)	1.9
T_F	Temperature of feed stream (°F)	190
T_1	Temperature in 1st effect (°F)	225
T_2	Temperature in 2nd effect (°F)	160
U_2	Heat-transfer coeff. in 2nd effect (Btu/min/ft^2/°F)	5.2345
W_1	Liquid holdup in 1st effect (lb)	30
W_2	Liquid holdup in 2nd effect (lb)	35
λ_S	Latent heat of condensation of steam (Btu/lb)	948

$$g_3(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \tag{44}$$

and g_{ij} denotes the jth component of the vector field $g_i(x)$.

Structural analysis and structural modification of the process

For the system of Eqs. 43 and 44, the relative order matrix has the form:

$$M_r = \begin{bmatrix} 2 & 1 & \infty \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \tag{45}$$

and the relative orders of the outputs y_1, y_2, y_3, with respect to the manipulated input vector u, are $r_1 = r_2 = r_3 = 1$. Furthermore, the corresponding characteristic matrix is:

$$C(x) = \begin{bmatrix} L_{g_1}h_1(x) & L_{g_2}h_1(x) & L_{g_3}h_1(x) \\ L_{g_1}h_2(x) & L_{g_2}h_2(x) & L_{g_3}h_2(x) \\ L_{g_1}h_3(x) & L_{g_2}h_3(x) & L_{g_3}h_3(x) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & g_{24}(x_2, x_3) & -1 \\ 0 & g_{25}(x_2, x_3, x_4, x_5) & 0 \end{bmatrix} \tag{46}$$

Clearly, rank $C(x) = 2 < 3$, and the matrix $C(x)$ is structurally singular due to its zero first column.

Following the structural modification algorithm described in the article, we first calculate the matrix $E^{(1)}(x)$ such that the last column of $C(x)E^{(1)}(x)$ becomes zero. In particular, choosing

$$E^{(1)}(x) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

we obtain:

$$C(x)E^{(1)}(x) = \begin{bmatrix} -1 & 0 & 0 \\ g_{24}(x_2, x_3) & -1 & 0 \\ g_{25}(x_2, x_3, x_4, x_5) & 0 & 0 \end{bmatrix}$$

Note that the above operation corresponds to a simple rearrangement of the input variables. The matrix $C(x)E^{(1)}(x)$ has only its first column ($l_1 = 1$) with two or more nonzero entries. An integrator is therefore added on the first input channel and an additional state variable $z_1 = u_2$ is defined.

With the extended state vector:

$$x^{(1)} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ z_1 \end{bmatrix}$$

the system in the extended state-space becomes:

$$\begin{aligned} \dot{x}_1 &= f_1^{(1)}(x_2, x_3, z_1) \\ \dot{x}_2 &= f_2^{(1)}(x_1, x_2, x_3) \\ \dot{x}_3 &= f_3^{(1)}(x_1, x_2, x_3) + g_{33}^{(1)}(x_1)v_3^{(1)} \\ \dot{x}_4 &= f_4^{(1)}(x_2, x_3, z_1) - v_2^{(1)} \\ \dot{x}_5 &= f_5^{(1)}(x_2, x_3, x_4, x_5, z_1) \\ \dot{z}_1 &= v_1^{(1)} \end{aligned} \tag{47}$$

where $v_1^{(1)}$, $v_2^{(1)}$, $v_3^{(1)}$ denote the redefined manipulated inputs with $v_1^{(1)} = \dot{u}_2$, $v_2^{(1)} = u_3$, $v_3^{(1)} = u_1$, and

$$f^{(1)}(x^{(1)}) = \begin{bmatrix} f_1^{(1)}(x_2, x_3, z_1) \\ f_2^{(1)}(x_1, x_2, x_3) \\ f_3^{(1)}(x_1, x_2, x_3) \\ f_4^{(1)}(x_2, x_3, z_1) \\ f_5^{(1)}(x_2, x_3, x_4, x_5, z_1) \\ f_6^{(1)} \end{bmatrix} = \begin{bmatrix} f_1(x_2, x_3) - z_1 \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \\ f_4(x_2, x_3) + g_{24}(x_2, x_3)z_1 \\ f_5(x_2, x_3, x_4, x_5) + g_{25}(x_2, x_3, x_4, x_5)z_1 \\ 0 \end{bmatrix} \tag{48}$$

$$g_1^{(1)}(x^{(1)}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad g_2^{(1)}(x^{(1)}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix};$$

$$g_3^{(1)}(x^{(1)}) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{\lambda_S}{x_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{49}$$

Referring to the extended system of Eq. 47, the relative order matrix is:

$$M_r^{(1)}=\begin{bmatrix}2 & \infty & 2\\ 2 & 1 & 2\\ 2 & 2 & 2\end{bmatrix} \tag{50}$$

and the relative orders of the outputs, y_1, y_2, y_3, with respect to the new manipulated input vector, are $r_1^{(1)}=2$, $r_2^{(1)}=1$, $r_3^{(1)}=2$.

Furthermore, the characteristic matrix takes the form:

$$C^{(1)}(x^{(1)})=\begin{bmatrix}L_{g_1^{(1)}}L_{f^{(1)}}h_1(x^{(1)}) & L_{g_2^{(1)}}L_{f^{(1)}}h_1(x^{(1)}) & L_{g_3^{(1)}}L_{f^{(1)}}h_1(x^{(1)})\\ L_{g_1^{(1)}}h_2(x^{(1)}) & L_{g_2^{(1)}}h_2(x^{(1)}) & L_{g_3^{(1)}}h_2(x^{(1)})\\ L_{g_1^{(1)}}L_{f^{(1)}}h_3(x^{(1)}) & L_{g_2^{(1)}}L_{f^{(1)}}h_3(x^{(1)}) & L_{g_3^{(1)}}L_{f^{(1)}}h_3(x^{(1)})\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{\partial f_1^{(1)}}{\partial x_6} & 0 & \dfrac{\lambda_S}{x_1}\dfrac{\partial f_1^{(1)}}{\partial x_3}\\ 0 & -1 & 0\\ \dfrac{\partial f_5^{(1)}}{\partial x_6} & -\dfrac{\partial f_5^{(1)}}{\partial x_4} & \dfrac{\lambda_S}{x_1}\dfrac{\partial f_5^{(1)}}{\partial x_3}\end{bmatrix} \tag{51}$$

which is generically nonsingular, as can be easily verified.

Controller synthesis and simulations

Because of the open-loop instability of the process (the two liquid holdups behave like simple integrators), the controller of theorem 5 was used in the simulations. To this end, following Daoutidis and Kravaris (1991), the following coordinate transformation was employed:

$$\zeta=T(x^{(1)})=\begin{bmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\\ \zeta_4\\ \zeta_5\\ \zeta_6\end{bmatrix}=\begin{bmatrix}t_1(x^{(1)})\\ h_1(x^{(1)})\\ L_{f^{(1)}}h_1(x^{(1)})\\ h_2(x^{(1)})\\ h_3(x^{(1)})\\ L_{f^{(1)}}h_3(x^{(1)})\end{bmatrix}=\begin{bmatrix}x_2\\ x_1\\ f_1^{(1)}(x_2,x_3,z_1)\\ x_4\\ x_5\\ f_5^{(1)}(x_2,x_3,x_4,x_5,z_1)\end{bmatrix} \tag{52}$$

to obtain the normal form description of the system of Eq. 47, and an explicit realization of the zero dynamics of the process. Requesting an input/output closed-loop decoupled response of the form:

$$y_1+\gamma_{11}^1\frac{dy_1}{dt}+\gamma_{12}^1\frac{d^2y_1}{dt^2}=y_{sp1}$$
$$y_2+\gamma_{21}^2\frac{dy_2}{dt}=y_{sp2}$$
$$y_3+\gamma_{31}^3\frac{dy_3}{dt}+\gamma_{32}^3\frac{d^2y_3}{dt^2}=y_{sp3} \tag{53}$$

the following controller realization was obtained from theorem 5:

$$u_1=v_3^{(1)}$$
$$u_2=z_1$$
$$u_3=v_2^{(1)}$$
$$\dot{\eta}_1=f_2^{(1)}(\eta_1,\mathfrak{D}y_1,\mathfrak{D}y_2,\mathfrak{D}y_3)$$
$$\dot{z}_1=v_1^{(1)}$$
$$v^{(1)}=\left(\begin{bmatrix}\beta_{12}^1 & 0 & 0\\ 0 & \beta_{21}^2 & 0\\ 0 & 0 & \beta_{32}^3\end{bmatrix}C^{(1)}(\eta_1,\mathfrak{D}y_1,\mathfrak{D}y_2,\mathfrak{D}y_3)\right)^{-1}$$
$$\times\begin{bmatrix}v_1-\sum_{k=0}^{1}\beta_{1k}^1\dfrac{d^ky_1}{dt^k}-\beta_{12}^1L_{f^{(1)}}^2h_1(x^{(1)})\\ v_2-y_2-\beta_{21}^2L_{f^{(1)}}h_2(x^{(1)})\\ v_3-\sum_{k=0}^{1}\beta_{3k}^3\dfrac{d^ky_3}{dt^k}-\beta_{32}^3L_{f^{(1)}}^2h_3(x^{(1)})\end{bmatrix}$$
$$v_1=\beta_{10}^1\xi_1^{(1)}+\left(\beta_{11}^1-\frac{\beta_{12}^1}{\gamma_{12}^1}\gamma_{11}^1\right)\xi_2^{(1)}+\frac{\beta_{12}^1}{\gamma_{12}^1}(y_{sp1}-y_1)$$
$$v_2=\beta_{20}^2\xi_1^{(2)}+\frac{\beta_{21}^2}{\gamma_{21}^2}(y_{sp2}-y_2)$$
$$v_3=\beta_{30}^3\xi_1^{(3)}+\left(\beta_{31}^3-\frac{\beta_{32}^3}{\gamma_{32}^3}\gamma_{31}^3\right)\xi_2^{(3)}+\frac{\beta_{32}^3}{\gamma_{32}^3}(y_{sp3}-y_3)$$
$$\dot{\xi}_1^{(1)}=\xi_2^{(1)}$$
$$\dot{\xi}_2^{(1)}=\frac{1}{\gamma_{12}^1}(y_{sp1}-y_1)-\frac{\gamma_{11}^1}{\gamma_{12}^1}\xi_2^{(1)}$$
$$\dot{\xi}_1^{(2)}=\frac{1}{\gamma_{21}^2}(y_{sp2}-y_2)$$
$$\dot{\xi}_1^{(3)}=\xi_2^{(3)}$$
$$\dot{\xi}_2^{(3)}=\frac{1}{\gamma_{32}^3}(y_{sp3}-y_3)-\frac{\gamma_{31}^3}{\gamma_{32}^3}\xi_2^{(3)} \tag{54}$$

The controller was tuned to give a slightly overdamped second-order closed-loop response for changes in y_{sp1} and y_{sp3} and a first-order response for changes in y_{sp2}. More specifically, the time constants and the damping factors for the three decoupled input/output responses were chosen as:

$$\tau_1=1.0\text{ min},\quad \zeta_1=1.1$$
$$\tau_2=4.4\text{ min}$$
$$\tau_3=2.236\text{ min},\quad \zeta_3=1.006$$

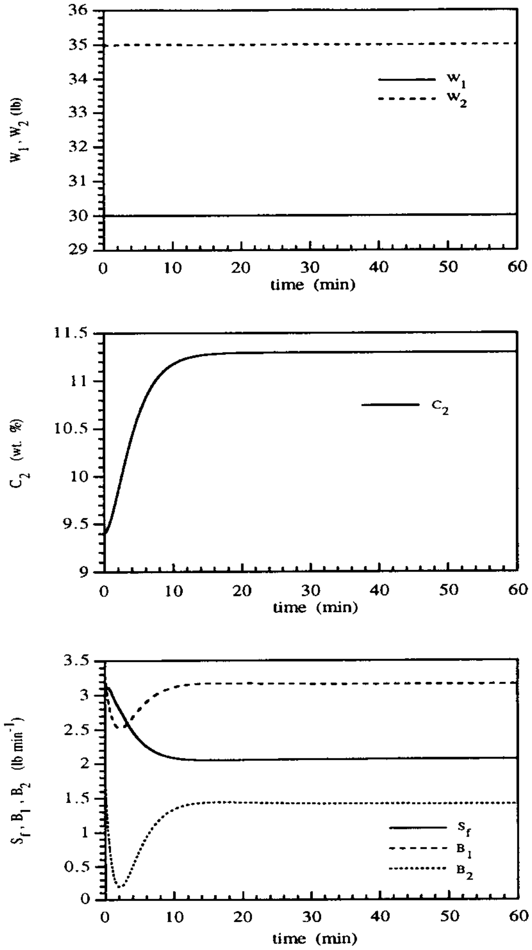

Figure 11. Closed-loop profiles for controlled outputs and manipulated inputs for a 20% increase in desired solute concentration C_2.

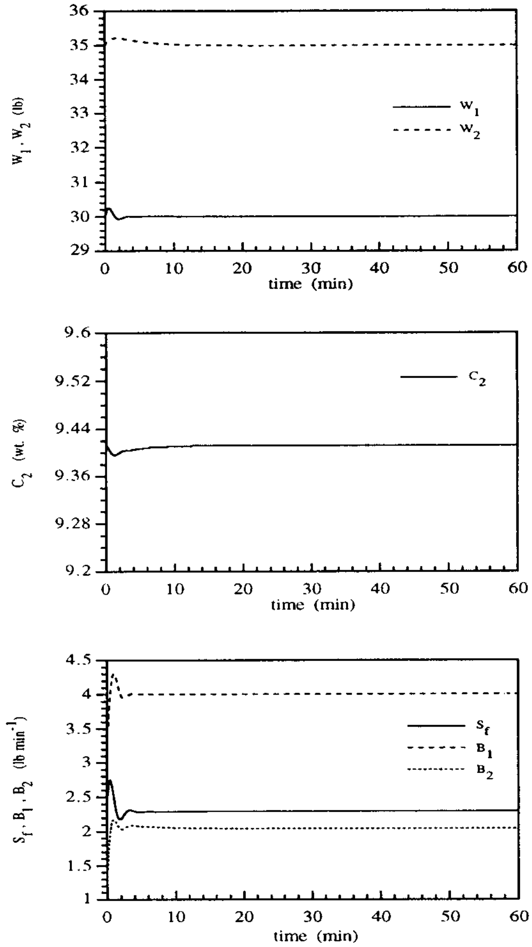

Figure 12. Closed-loop profiles for controlled outputs and manipulated inputs for a 20% increase in feed flow rate F.

through the following choice of the controller parameters:

$$\beta_{10}^{(1)}=6, \quad \beta_{11}^{(1)}=2.2, \quad \beta_{12}^{(1)}=1.0$$

$$\beta_{20}^{(2)}=5, \quad \beta_{21}^{(2)}=4.4$$

$$\beta_{30}^{(3)}=20, \quad \beta_{31}^{(3)}=5, \quad \beta_{32}^{(3)}=3$$

$$\gamma_{11}^{(1)}=2.2, \quad \gamma_{12}^{(1)}=1.0$$

$$\gamma_{21}^{(2)}=4.4$$

$$\gamma_{31}^{(3)}=4.5, \quad \gamma_{32}^{(3)}=5.0$$

Several simulations were performed to evaluate the set-point tracking and disturbance rejection capabilities of the nonlinear controller, as well as its robustness characteristics with respect to initialization errors and modeling errors. The performance of the nonlinear controller was also compared with that of a Linear Quadratic Regulator (LQR). In all the simulation runs, the process was assumed to be at its nominal steady state, and the set point and/or disturbance changes were imposed at time $t=0$.

The first simulation run addressed the set point tracking capabilities of the controller for a 20% increase in the solute concentration C_2 in the product stream, that is, y_{sp3}. Figure 11 shows the profiles of the three controlled outputs and the manipulated inputs. One can readily observe the decoupled responses of y_1 and y_2, and the theoretically predicted response of y_3.

The subsequent three runs addressed the disturbance rejection capabilities of the controller. More specifically, the following disturbances were considered:

- 20% increase in the feed flow rate F;
- 20% increase in the solute concentration in the feed stream, C_F; and
- 20% increase in the specific enthalpy of the feed stream, h_F.

Figures 12, 13, and 14 illustrate the profiles of the controlled outputs and the manipulated inputs for each of the above cases, respectively. As can be seen, in all three cases the controller exhibits excellent regulatory characteristics.

The next run was carried out to demonstrate the set point tracking capabilities of the controller in the presence of a

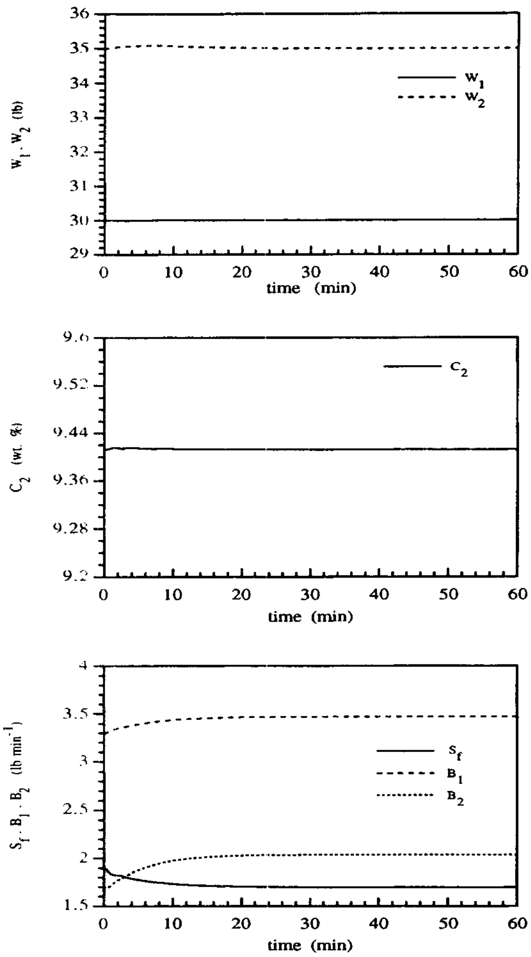

Figure 13. Closed-loop profiles for controlled outputs and manipulated inputs for a 20% increase in solute concentration C_F in the feed stream.

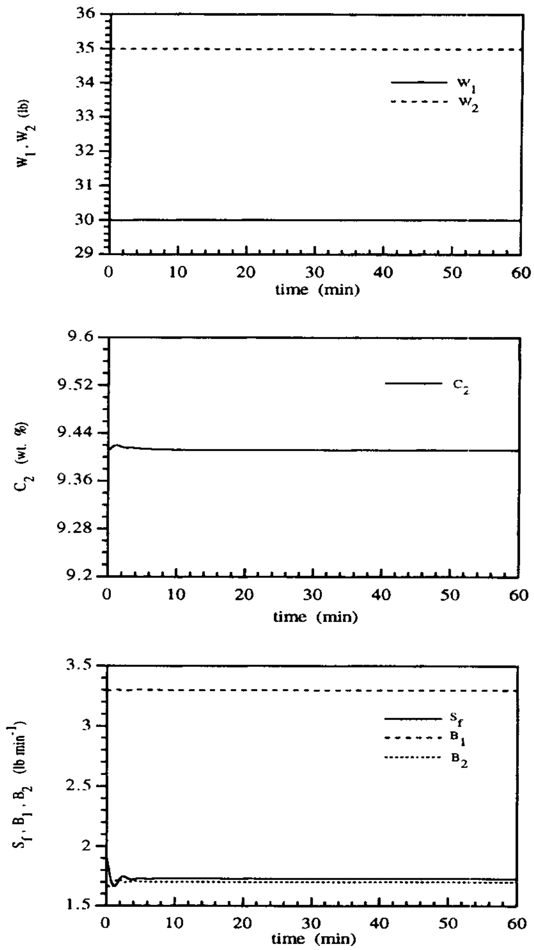

Figure 14. Closed-loop profiles for controlled outputs and manipulated inputs for a 20% increase in specific enthalpy h_F in the feed stream.

disturbance when both the desired solute concentration C_2 and the feed flow rate F were increased simultaneously by 20%. Figure 15 illustrates the responses for the three controlled outputs. Due to the unmeasured disturbance, the first two outputs (W_1 and W_2) show slight deviations initially, while the third output (C_2) follows a second-order response very close to the theoretically predicted one.

The next set of simulation runs tested the performance of the controller in the presence of initialization errors and parametric uncertainty. Initially, a 20% increase in y_{sp3} was imposed under a 5% initialization error in the controller state η_1 and a 10% error in the overall heat-transfer coefficient U_2. As can be seen from Figure 16, the controller performs very satisfactorily in maintaining the first two outputs at their set points, whereas the response of y_3 is very close to the one predicted by the theory. In the next run, a 20% increase in the feed flow rate F was imposed under the same initialization and modeling errors as before. Figure 17 shows the very satisfactory response of the three controlled outputs and the corresponding manipulated inputs. The set-point tracking and disturbance rejection capabilities of the nonlinear controller were also compared with those of a standard LQR. The following weighting matrices were chosen through trial and error for the linear controller design:

$$Q = \text{diag}(10, 1, 1, 10, 10, 0.01, 0.01, 10)$$

$$R = \text{diag}(1, 1, 1)$$

to obtain good responses with reasonable control effort.

Figure 18 shows the responses for the nonlinear controller and the LQR for a 20% increase in the desired solute concentration C_2. While the nonlinear controller exhibits a totally decoupled response, the LQR shows significant deviations in y_1 and y_2. Moreover, y_3 shows a better response under the nonlinear controller compared to that with the LQR which exhibits oscillations. Finally, the performance of the nonlinear controller was compared with that of the LQR in rejecting the effects of a 20% increase in the feed flow rate F. The profiles of the three controlled outputs and the corresponding manipulated inputs are shown in Figure 19, and clearly illustrate the superior performance of the nonlinear controller.

Acknowledgment

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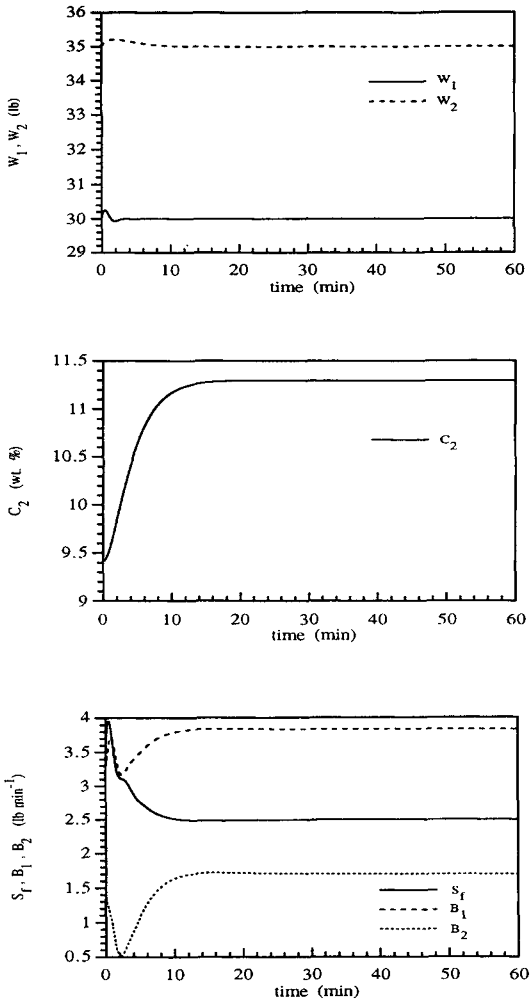

Figure 15. Closed-loop profiles for controlled outputs and manipulated inputs for a 20% increase in desired solute concentration C_2 in the presence of a 20% increase in feed flow rate F.

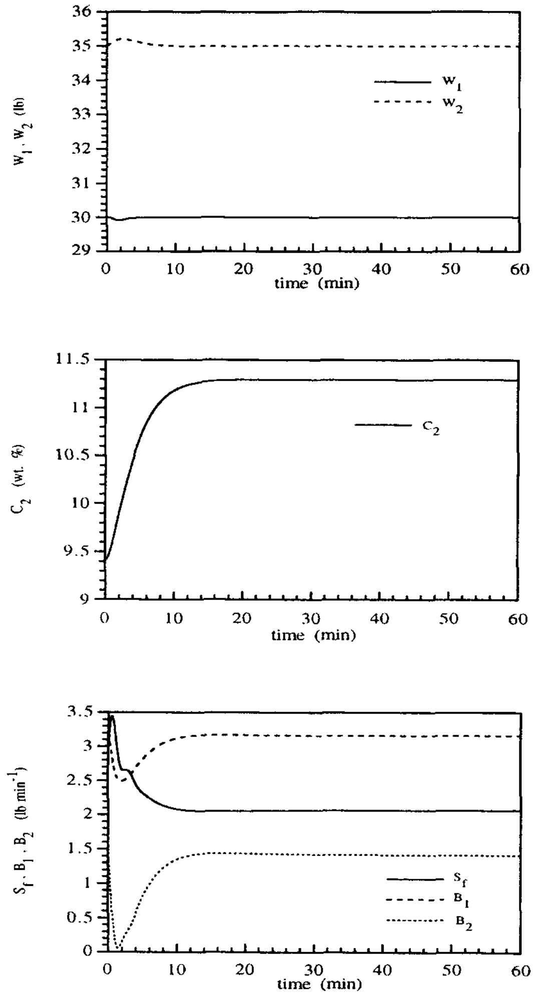

Figure 16. Closed-loop profiles for controlled outputs and manipulated inputs for a 20% increase in desired solute concentration C_2 under initialization and modeling errors.

Notation

C = characteristic matrix
E = set of edges in a bipartite graph
$E^{(\kappa)}$ = feedback matrix that localizes the singularity of the characteristic matrix
f = vector field
g_j = vector field associated with u_j
G = bipartite graph
h_i = scalar field associated with y_i
r_{ij} = relative order of y_i with respect to u_j
r_i = relative order of y_i with respect to u
s = the Laplace domain variable
s_i = indices denoting order of closed-loop response
t = time
u = manipulated input vector
U = set of vertices of manipulated inputs
v = auxiliary input vector
w = state vector of process model
x = state vector of process
y_i = process output
Y = set of vertices of controlled outputs
y_{spi} = output set point
z = state variables of dynamic state feedback

Greek letters

β^j_{ik} = adjustable parameters
γ^j_{ik} = adjustable parameters
ζ = state vector in normal form coordinates
η = controller state variables
ξ = state variables of linear compensator

Math symbols

det = determinant of a matrix
diag = diagonal matrix
$\not\equiv$ = not equivalently equal to
$\mathbb{R}^n$ = n-dimensional Euclidean space
T = transpose
$|\cdot|$ = cardinality of a set

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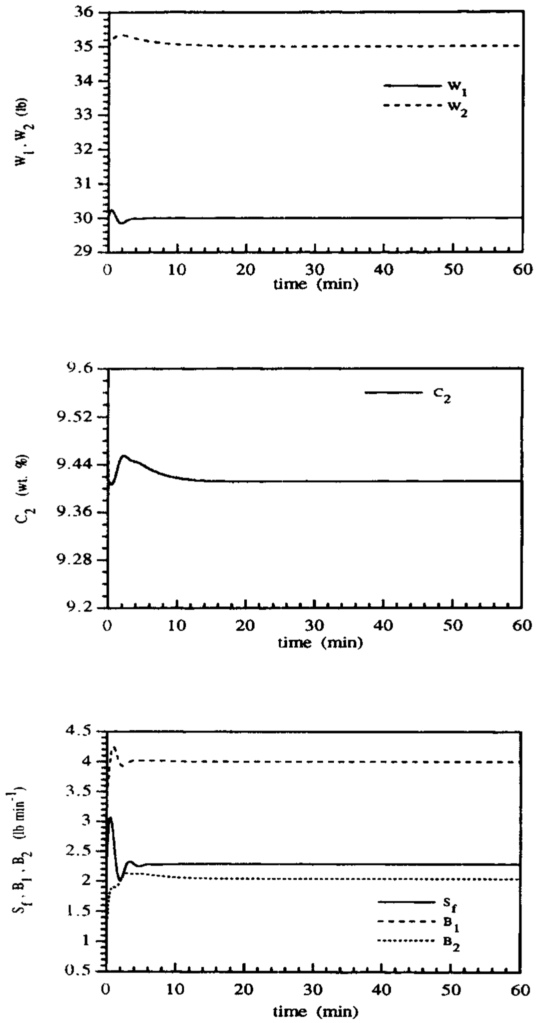

Figure 17. Closed-loop profiles for controlled outputs and manipulated inputs for a 20% increase in feed flow rate F under initialization and modeling errors.

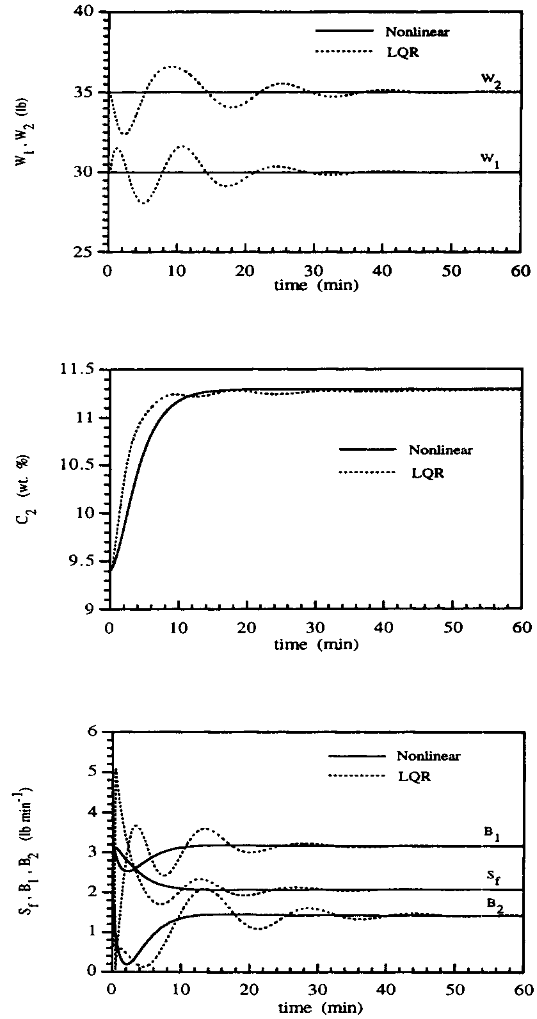

Figure 18. Comparison of closed-loop responses to a 20% increase in desired solute concentration C_2 under LQR and nonlinear controller.

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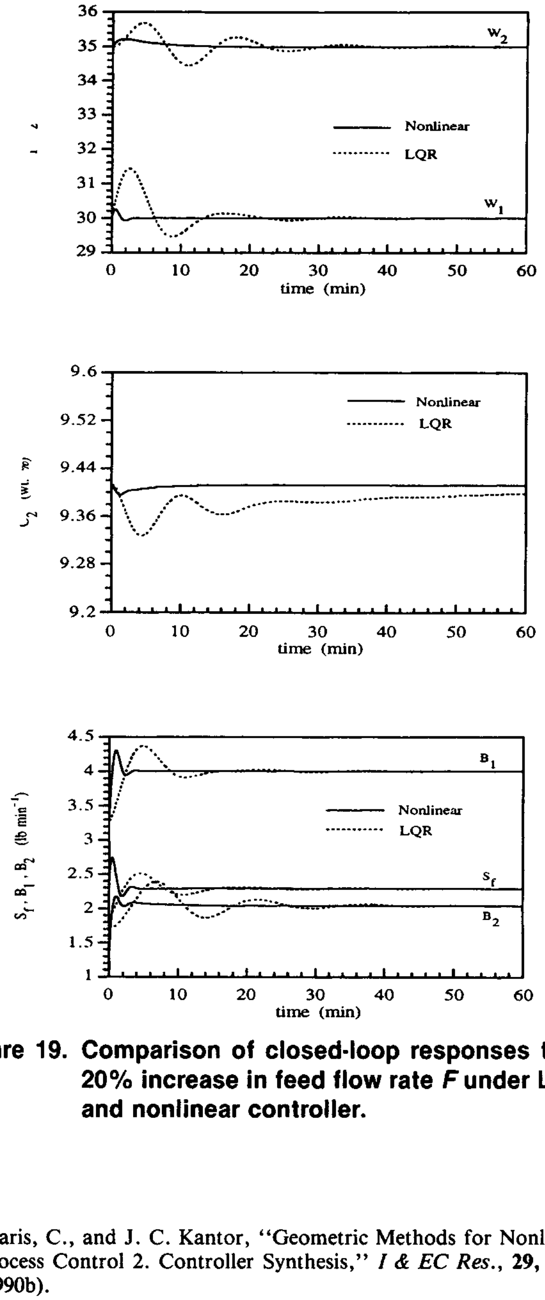

Figure 19. Comparison of closed-loop responses to a 20% increase in feed flow rate F under LQR and nonlinear controller.

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Appendix

Proof of theorem 2

We will first need some additional concepts from graph theory. In particular, a *matching* M in a bipartite graph is a set of edges with no common endpoints. A matching M in the bipartite graph $G(A, B, E)$ is then called an *A-matching* if $|M| = |A|$. We will also need the following classical theorem of graph theory.

Hall's Marriage Theorem. A bipartite graph $G(A, B, E)$ has an A-matching if and only if for each subset $X \subseteq A$, $|R(X)| \geq |X|$, where $R(X)$ is the set of vertices in the set B which are adjacent to vertices in the set X.

We can now proceed with the proof of the theorem.

Proof of 1⇒2. Suppose that statement 1 of theorem 2 holds, that is, the generic rank of the structural matrix equivalent to $C(x)$ is equal to m. Then, according to theorem 1, we can rearrange the output variables so that the relative order matrix M_r takes the form of Eq. 9, that is, each pair $\{y_i, u_i\}$ is such that $r_{ii} = r_i$. Hence, the bipartite graph $G(U, Y, E)$ defined in theorem 2 has a Y-matching, and by Hall's theorem, statement 2 of theorem 2 also holds.

Proof of 2⇒1. Suppose that statement 2 of theorem 2 holds. Then, by Hall's theorem, the bipartite graph $G(U, Y, E)$ defined in theorem 2 has a Y-matching, that is, for each output y_i we can find a distinct input u_j such that $r_{ij} = r_i$. Thus, we can rearrange the outputs y_i so that the relative order matrix M_r takes the form of Eq. 9. Hence, according to theorem 1, the structural matrix equivalent to $C(x)$ has generic rank equal to m and statement 1 of theorem 2 holds.

Proof of theorem 3

Consider the nonlinear process described by Eq. 1. The cascade of dynamic compensators applied to the process during the structural modification algorithm has the form:

$$
\begin{aligned}
u &= \sum_{j=1}^{l_1} e_j^{(1)}(x) z_j + \sum_{j=l_1+1}^{m} e_j^{(1)}(x) v_j^{(1)} \\
\dot z_1 &= v_1^{(1)} \\
\dot z_2 &= v_2^{(1)} \\
&\vdots \\
\dot z_{l_1} &= v_{l_1}^{(1)} \\
v^{(1)} &= \sum_{j=1}^{l_2} e_j^{(2)}(x^{(1)}) z_{l_1+j} + \sum_{j=l_2+1}^{m} e_j^{(2)}(x^{(1)}) v_j^{(2)} \\
\dot z_{l_1+1} &= v_1^{(2)} \\
&\vdots \\
\dot z_{l_1+l_2} &= v_{l_2}^{(2)} \\
&\vdots \\
v^{(\kappa-1)} &= \sum_{j=1}^{l_\kappa} e_j^{(\kappa)}(x^{(\kappa-1)}) z_{l_1+\ldots+l_{\kappa-1}+j} \\
&\quad + \sum_{j=l_\kappa+1}^{m} e_j^{(\kappa)}(x^{(\kappa-1)}) v_j^{(\kappa)} \\
\dot z_{l_1+\ldots+l_{\kappa-1}+1} &= v_1^{(\kappa)} \\
&\vdots \\
\dot z_{l_1+\ldots+l_\kappa} &= v_{l_\kappa}^{(\kappa)}
\end{aligned}
\tag{55}
$$

Considering the resulting system of Eq. 23, it is straightforward to verify (for example, Kravaris and Soroush, 1990) that the control law:

$$v^{(\kappa)} = \{[\beta_{1r_1^{(\kappa)}} \ldots \beta_{mr_m^{(\kappa)}}]C^{(\kappa)}(x^{(\kappa)})\}^{-1} \times \left[v - \sum_{i=1}^{m} \sum_{k=0}^{r_i^{(\kappa)}} \beta_{ik} L_{f(\kappa)}^{k} h_i(x) \right] \quad (56)$$

induces an input/output behavior of the form:

$$\sum_{i=1}^{m} \sum_{k=0}^{r_i^{(\kappa)}} \beta_{ik} \frac{d^k y_i}{dt^k} = v$$

Combining the control law of Eq. 56 with the dynamic state feedback compensators of Eq. 55 results in the controller of Eq. 27 which clearly induces the input/output behavior of Eq. 26.

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